On Killing Vector Fields of Particular Lorentzian Metrics with Applications to General Relativity

Diploma Thesis

written by
Jörg Christian Reiher

supervised by
**Priv.-Doz. Dr. Mike Scherfner**
Department of Mathematics
Faculty II, Technische Universität Berlin

reviewed by
**Prof. Dr. Wolfgang K. Schief**
Department of Mathematics
Faculty II, Technische Universität Berlin

## Preface

I would like to thank my supervisor, Dr. Mike Scherfner, for suggesting a topic that suits my mathematical field of interest so well and for his constant support. I would also like to thank Prof. Dr. Wolfgang K. Schief for reviewing this work.

Having focused on Riemannian geometry during my studies, I quickly realized that general relativity has many aspects that cannot be explained solely with mathematics. Matthias Plaue and Alexander Dirmeier always helped me with my questions on such aspects, which I am very grateful for. I am also very grateful to Dr. Günter Paul Peters, for his advice on the application of concepts I had learned from him in my differential geometry courses. This work uses results from a paper by Dr. Thoralf Chrobok, whom I would like to thank for his advice and encouragement.

My further thanks go to my fellow students David Chubelaschwili for his encouragement when things were challenging and Dario Götz for his support whenever I was struggling with mathematical software.
Dr. Katharina Zeitz and Dr. Philip Zeitz supported me with valuable advice, especially towards the end of my work, which I am very grateful for.

My special thanks go to Prof. Dr. Dirk Ferus and all lecturers at Technische Universität Berlin and Durham University for their guidance throughout my studies.

On a final note, this work is dedicated to my former mathematics teacher at Kant Gymnasium Berlin, Studienrat Hans-Joachim Brehm, who awakened at an early stage my passion for mathematics.

# Contents

# Introduction

The mathematical foundations of general relativity theory are the concepts of semi-Riemannian geometry. In order to describe cosmological models one uses 4-dimensional semi-Riemannian manifolds. These models must have certain symmetries to be physically viable.
Such symmetries are represented by Killing vector fields on manifolds.
An important class of cosmic models are the Gödel type solutions. In 1949 Gödel published a metric that models a rotating universe with vanishing shear and acceleration as well as vanishing expansion. The Killing vectors of this metric are well known.
In 2009 M. Gürses, M. Plaue, M. Scherfner, T. Schönfeld and L. A. M. de Sousa published two generalizations of the Gödel metric.
The Killing vectors of these generalized metrics have not been calculated so far.

The aim of this work is to introduce the mathematical concepts of Killing vector fields and their role in general relativity theory to a reader with basic knowledge in differential geometry. These concepts are then applied to calculate the Killing vectors of the generalized Gödel metrics mentioned above.
Chapter 1 contains a short repetition of the basics in semi-Riemannian geometry including sections on tensor fields and Lie algebras. These concepts are fundamental for a profound understanding of the material. Section 1.4 shortly explains how semi-Riemannian geometry is applied in general relativity theory.
In Chapter 2 the definition of a Killing vector field is introduced followed by important theorems and examples. Then three sections explain how 2-, 3- and 4-dimensional Lie algebras such as Killing vector fields can be classified. This turns out to be of great importance in Chapter 3 where the relevance of Killing vector fields to several concepts in general relativity theory is explained in detail.
In Chapter 4 an approach by T. Chrobok to calculate the Killing vectors of the Gödel metric is explained carefully. Then a slightly different approach on how to calculate the Killing vectors of the generalized Gödel metrics mentioned above is presented in detail.
The appendix contains a commented maple® worksheet that has been used for the calculations in Sections 4.2 and 4.3.

## Historical note

Wilhelm Carl Joseph Killing was born on May 10th 1847 in Burbach. He commenced his studies in mathematics in 1865 at the University of Münster, but went to Berlin in 1867 to continue as a student of Weierstraß. In 1872 he finished his dissertation "Die Flächenbündel zweiter Ordnung". This work was an important foundation for Killings later studies on the classification of Lie algebras. In 1881 Killing became professor at the University of Braunsberg where he continued his studies in Geometry under adverse conditions. The library in Braunsberg was very limited and Killing had a lot of administrative duties. Nevertheless Killing published his work "Erweiterung des Raumbegriffs" in 1884 in which he classifies all Lie algebras of dimension 2 and 3. Lie had made important advances in this field already in 1873. Killing discovered his results independently of Lie since his library did not contain the Scandinavian journal in which Lie's article appeared. When Killing found out about Lie's work he even had difficulties getting in touch with Lie on that topic. He had to convince Lie that he was only interested in the geometric aspects of Lie's work before he obtained a copy of it. In 1892 Killing finished his work "Über die Grundlagen der Geometrie" which contained the concepts of what today is called a "Killing vector field". Shortly after that he became professor at the University of Münster. Five years later he was honoured with the Lobaschewsky prize (as Lie had been in 1897). Killing died in 1923 in Münster.

# Chapter 1

# Preliminaries

Before dealing with Killing vector fields we will shortly go through the basics of semi-Riemannian geometry, i.e. basic notation for manifolds, smooth maps and vector fields. Afterwards we will do the same for tensors, and the basics of the theory of Lie groups. These topics provide the basic concepts to understand Killing vector fields. Section 1, 2 and 3 follow the concept of the according chapters in [13]. The approach in this book provides a good geometric intuition since it is mainly coordinate free. Section 1.4 gives an overview on how differential geometry is applied in general relativity theory. These are only a few motivations for the basic definitions. A detailed introduction to general relativity theory can be found in [18].

## 1.1 Semi-Riemannian Geometry

**Remark 1.1.** "$n$-dimensional Manifold"
*If not mentioned otherwise we will be dealing with* ***differentiable manifolds*** *that are Hausdorff spaces and fulfil the second axiom of countability.*
*Manifolds are denoted by $M$ with differentiable charts*
$X_\alpha : M \supset U_\alpha \mapsto \mathbb{R}^n$, *$\alpha \in I$ with an index set $I$.*
*Every chart has components $x^i_\alpha : M \supset U_\alpha \mapsto \mathbb{R},\ i \in \{1, ..., n\}$.*
*In those cases where we deal with local properties we will suppress the index $\alpha$ leaving us with a chart $X$ and its components $x^i$. The* ***tangent space*** *at a point $p \in M$ is denoted by $T_pM$.*

**Definition 1.1.** "Inner Product"
*An* ***inner product*** *on a $n$-dimensional vector space $\tilde{V}$ is a non-degenerate, symmetric bilinear form $g : \tilde{V} \times \tilde{V} \mapsto \mathbb{R}$.*
*With respect to (w.r.t.) an appropriate orthonormal basis $\{e_1, ..., e_n\} \subset \tilde{V}$*

*the representative matrix of $g$ with components $g_{ij} = g(e_i, e_j)$ is of the form*

$$g_{ij} = \epsilon_i \delta_i^j, \ i, j \in \{1, ..., n\}, \ \epsilon \in \{1, -1\}.$$

*If not mentioned otherwise, $\delta_i^j$ always denotes the **Kronecker symbol**. Linear algebra tells us that the signature of $\epsilon_i$ does not depend on the chosen basis and therefore is an invariant of $g$. One defines the **signature** of $(\tilde{V}, g)$ as*

$$(m, n - m),$$

*with $n := dim(\tilde{V})$, $m := \#\{i \mid \epsilon_i < 0\}$.*
*By $\mathbb{R}_m^n$ we will denote the $n$-dimensional real vector space with signature $(m, n-m)$. For reasonable small $n$ one can also write*

$$(\underbrace{-...-}_{n}\underbrace{+...+}_{n-m}).$$

*E.g. $(-+++)$ in $\mathbb{R}^4$ with $m = 1$.*
*The diagonal matrix $S$ with components $s_{ij} = \epsilon_i \delta_i^j$ for $i, j \in \{1, ..., n\}$ is called the **signature matrix** of a basis in $(\tilde{V}, g)$.*

For the coordinate vectors $y, w \in \mathbb{R}^n$ of two vectors $Y, W \in \tilde{V}$ w.r.t an orthonormal basis this implies

$$g(Y, W) = y^T S w.$$

**Definition 1.2.** "Semi-Riemannian Manifold"
*A **metric tensor** is a positive definite, non-degenerate, symmetric 2-form $g$ on $M$. In other words: A metric $g$ smoothly assigns to every point $p \in M$ an inner product $g_p$ on $T_pM$ (Notation for tensors will be explained in the next section). Its components are referred to by $g_{ij}$, and those of its inverse by $g^{ij}$.*

- *A manifold $M$ together with a metric $g$ is called a **Riemannian manifold** denoted by $(M, g)$.*
- *If $g_p$ is semi-definite at a point $p \in M$ (i.e. $\forall V_p \in T_pM : g(V_p, V_p) \geq 0$) then $(M, g)$ is called a **semi-Riemannian manifold**.*
- *If $g$ has signature $(1, n-1)$ then $(M, g)$ is called a **Lorentzian manifold**.*

*From now on, if not mentioned otherwise, $M$ will always denote a $n$-dimensional semi-Riemannian manifold with a metric $g$.*

**Remark 1.2.**
*The following facts and definitions will be used frequently.*

- *If not mentioned otherwise sums always run over the whole index range. Usually this will be* $\{1, ..., n\}$.
- *The* ***tangent space*** *of* $M$ *at a point* $p \in M$ *is denoted by* $T_pM$. *The directional derivative of a smooth function* $f \in C^\infty(M)$, $f : M \mapsto \mathbb{R}$ *on* $M$ *along* $V_p \in T_pM$ *is denoted by*

  $$V_p f.$$

- *The* ***tangent bundle*** *of* $M$ *is denoted by* $TM$ *such that*

  $$TM := \bigcup_{p \in M} (T_pM).$$

- *The set of all* ***vector fields*** *on* $M$ *is denoted by* $\Gamma(TM)$. *For a fixed coordinate chart* $X_\alpha$ *and* $i \in \{1, ..., n\}$ *one defines*

  $$\delta_i|_p := \frac{\partial}{\partial x^i}|_p.$$

  *This implies* $Vf = \sum_i V^i \delta_i f$, $V^i \in C^\infty(M)$ *for* $i \in \{1, ..., n\}$. *Along with the basis* $\{\delta_i|_p\}_{i\in\{1,...,n\}}$ *of* $(T_pM, g_p)$ *comes its* ***dual basis***

  $$\{dx^i|_p\}_{i\in\{1,...,n\}}.$$

- *Maps and functions will usually be smooth. Notation:* $\phi : M \mapsto N$, $p \to \phi(p)$, *where* $M$ *and* $N$ *are manifolds. The* ***differential map*** *is denoted by* $D\phi : TM \mapsto TN$, $V_p \to D\phi(V_p)$ *and one defines for* $V_p \in T_pM$ *and* $f \in C^\infty(N)$:

  $$D\phi(V_p)f := V_p(f \circ \phi).$$

- *For a vector field* $V \in \Gamma(TM)$, $p \in M$ *the (local)* ***flow*** $\phi$ *is defined as*

  $$\phi : M \times \mathbb{R} \supset (a, b) \mapsto M, \phi_t(p) = \gamma_p(t)$$

  *with* $\gamma_p : \mathbb{R} \mapsto M$ *being the maximal integral curve of* $V$ *through* $p$ *(i.e.* $\gamma : \mathbb{R} \supset (-\epsilon, \epsilon) \mapsto M$, $\gamma(0) = p$, $\gamma'(t) = V_{\gamma(t)}$*).*
- *A vector field is* ***complete*** *if all of its integral curves are defined on the whole real line* $\mathbb{R}$.

**Definition 1.3.** "Lie Bracket"
*The **Lie bracket** is a map* $[.,.] : \Gamma(TM) \times \Gamma(TM) \mapsto \Gamma(TM)$ *such that*

$$[V, W](f) := VWf - WVf$$

*for all* $f \in C^\infty(M),\ V, W \in \Gamma(TM)$.

Schwarz's theorem yields: $[\delta_i, \delta_j] = 0$.

**Lemma 1.1.** "Properties of the Lie Bracket"
*For* $V, W, Y, Z \in \Gamma(TM)$ *the Lie Bracket has the following properties:*

1. *The Lie bracket is* $\mathbb{R}$*-linear,*
2. *Skew-symmetry:* $[W, V] = -[V, W]$,
3. *Jacobi identity:* $[V, [Y, Z]] + [Y, [Z, V]] + [Z, [V, Y]] = 0$.

*Proof.*
The proof can be found in [13, p. 13]. □

The Lie bracket does not necessarily have to be connected with the curvature of a manifold. It only takes into account how two vector fields behave compared to each other. Connecting the behaviour of the Lie bracket with the curvature of a manifold is one of the ideas behind the definition of a covariant derivative.

**Definition 1.4.** "Connection"
*A **connection*** $\nabla$ *on* $M$ *is a map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \mapsto \Gamma(TM), (V, W) \mapsto \nabla_V W$$

*such that for* $V, W, Y \in \Gamma(TM),\ f \in C^\infty(M)$*:*

1. $\nabla_V W$ *is* $C^\infty(M)$*-linear in* $V$*,*
2. $\nabla_V (fW) = (Vf)W + f(\nabla_V W)$.

*A connection is called **Levi-Civita connection** (subsequently abbreviated with "L.-C. connection") if*

1. $\nabla$ *is torsion-free:*
   $[V, W] = \nabla_V W - \nabla_W V$,

2. *$\nabla$ is compatible with the metric $g$:*
   $Yg(V,W) = g(\nabla_Y V, W) + g(V, \nabla_Y W)$.

It follows from the definition that the L.-C. connection is unique on $(M,g)$. From now on, if not mentioned otherwise, $M$ will always denote a semi-Riemannian manifold with a metric $g$ and its L.-C. connection $\nabla$.

**Definition 1.5.** "Christoffel Symbol"
*Along with the L.-C. Connection come the Christoffel symbols $\Gamma_{ij}^k \in C^\infty(M)$, defined by:*

$$\nabla_{\delta_i}\delta_j = \sum_k \Gamma_{ij}^k \ \delta_k, \ i \in \{1, ..., n\}.$$

**Definition 1.6.** "Riemannian Curvature Tensor"
*The **Riemannian Curvature Tensor** is a map*
$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \mapsto \Gamma(TM)$,

$$R_{VW}Z := \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z.$$

$V, W, Z \in \Gamma(TM)$. *See also Example 1.1.*

**Remark 1.3.** "Notation"
*Coordinates of tangent vectors are denoted by an upper index:*

$$V = \sum_i V^i \delta_i, \ V \in \Gamma(TM), \ V^i \in C^\infty(M) \ for \ i \in \{1, ..., n\}.$$

*Coordinates of one-forms are denoted by a lower index:*

$$\alpha = \sum_i \alpha_i dx^i, \ \alpha \in \Gamma(TM^*), \ \alpha_i \in C^\infty(M) \ for \ i \in \{1, ..., n\}.$$

*The L.-C. Connection is also called **covariant derivative**. Covariant derivatives are denoted by a semicolon:*

$$\nabla_{\delta_j} V = \sum_k V_{;j}^k \delta_k, \ V \in \Gamma(TM), \ V_j^k \in C^\infty(M) \ for \ k, j \in \{1, ..., n\}.$$

*Partial derivatives are denoted by a comma:*

$$V^i_{,j} := \frac{\partial V^i}{\partial x^j},\ V \in \Gamma(TM).$$

*Furthermore the definition of the Christoffel symbols yields:*

$$V^i_{;j} = V^i_{,j} + \sum_k V^k \Gamma^i_{jk},\ V \in \Gamma(TM).$$

*For a vector field $V \in \Gamma(TM)$ it is not clear whether $V_p$ denotes the value of $V$ at a point $p \in M$ or the pth covariant component of $V$. Therefore $p$ and $q$ as subscripts in the form $V_p$ and $V_q$ shall always denote points on $M$ such that $p, q \in M$, $V_p \in T_pM$, $V_q \in T_qM$. Since coordinate free notation usually is not mixed with local coordinates this should not cause any problems.*

**Definition 1.7.** "Geodesic"
*A curve $\gamma : \mathbb{R} \supset (a,b) \to M$ is a* ***geodesic*** *if*

$$\nabla_{\gamma'}\gamma' = 0. \tag{1.1}$$

*This means that the vector field $\frac{d}{dt}\gamma = \gamma'$ does not change along $\gamma$.*

Furthermore for every point $p \in M$ one can find a neighbourhood $U \subset T_pM$ such that for every $V_p \in U$ there exists a geodesic $\gamma_{V_p} : [0,1] \mapsto M$ which satisfies $\gamma_{V_p}(0) = p$ and $\gamma'_{V_p}(0) = V_p$. $\gamma_{V_p}$ is locally the unique geodesic through $p$ in direction of $V_p$. A proof for this statement can be found in [13, p. 72].

**Definition 1.8.** "Geodesic Variation"
*A smooth map $h : \mathbb{R}^2 \supset (a,b) \times (c,d) \to M,\ (s,t) \mapsto h(s,t)$ is a* ***geodesic variation*** *if for every $s \in (a,b)$*

$$h(s,\cdot) =: \gamma_s(.)$$

*is a geodesic. The vector field $\frac{\partial}{\partial s}h(0,\cdot)$ is called the* ***variation vector field*** *of $h$.*

**Definition 1.9.** "Exponential Map"
*Choosing* $U \subset T_pM$ *as in Definition 1.7 one defines:*

$$exp_p : T_pM \to M,\ exp_p(V_p) := \gamma_{V_p}(1),\ V_p \in T_pM.$$

$exp_p$ *is the* ***exponential map*** *at* $p$ *and* $U_p := exp_p(U)$ *is called a* ***normal neighborhood of*** $p$.

The fact that $Dexp_p(V_0) = V_0$ for all $V_0 \in T_pM$ together with the inverse function theorem yield that $exp_p$ is a **local diffeomorphism** at every point $p \in M$.

**Definition 1.10.** "Jacobi Field"
*A vector field* $Y \in \Gamma(TM)$ *along a geodesic* $\gamma$ *on* $M$ *is called a* ***Jacobi field*** *if it satisfies*

$$Y'' + R_{Y\gamma'}(\gamma') = 0. \tag{1.2}$$

In the context of Killing vector fields, Jacobi fields are of special interest as we will see later.

**Lemma 1.2.**
*The variation vector field of a geodesic variation on* $M$ *is a Jacobi field.*

*Proof.*
Let $h(s,t)$ be a geodesic variation of the geodesic $\gamma(t)$, i.e. $h(0,t) = \gamma(t)$. The geodesic equation (1.1) yields:

$$\nabla_{\frac{\partial h}{\partial t}} \frac{\partial h}{\partial t} = 0 \Rightarrow \nabla_{\frac{\partial h}{\partial s}} \nabla_{\frac{\partial h}{\partial t}} \frac{\partial h}{\partial t} = 0. \tag{1.3}$$

As we have already seen the curvature Tensor $R$ obeys:

$$R_{\frac{\partial h}{\partial t},\frac{\partial h}{\partial s}} \frac{\partial h}{\partial t} = \nabla_{\frac{\partial h}{\partial t}} \nabla_{\frac{\partial h}{\partial s}} \frac{\partial h}{\partial t} - \nabla_{\frac{\partial h}{\partial s}} \nabla_{\frac{\partial h}{\partial t}} \frac{\partial h}{\partial t} - \nabla_{[\frac{\partial h}{\partial t},\frac{\partial h}{\partial s}]} \frac{\partial h}{\partial t}.$$

Together with (1.3) this yields:

$$0 = \nabla_{\frac{\partial h}{\partial s}} \nabla_{\frac{\partial h}{\partial t}} \frac{\partial h}{\partial t} = \nabla_{\frac{\partial h}{\partial t}} \nabla_{\frac{\partial h}{\partial s}} \frac{\partial h}{\partial t} - R_{\frac{\partial h}{\partial t}\frac{\partial h}{\partial s}} \frac{\partial h}{\partial t}. \tag{1.4}$$

Since $h$ is a geodesic variation and $\nabla$ is torsion-free we have:

$$0 = [\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}] \Rightarrow \nabla_{\frac{\partial h}{\partial s}} \frac{\partial h}{\partial t} = \nabla_{\frac{\partial h}{\partial t}} \frac{\partial h}{\partial s}.$$

Thus from (1.4) one obtains the Jacobi field equation with $Y = \frac{\partial h}{\partial s}|_0$:

$$
\begin{aligned}
0 &= \nabla_{\frac{\partial h}{\partial t}|_0} \nabla_{\frac{\partial h}{\partial t}|_0} \frac{\partial h}{\partial s}|_0 - R_{\frac{\partial h}{\partial t}|_0, \frac{\partial h}{\partial s}|_0} \frac{\partial h}{\partial t}|_0 \\
&= Y'' + R_{Y\gamma'}\gamma'.
\end{aligned}
$$

□

**Lemma 1.3.** "Uniqueness Conditions for a Jacobi Field"
*Let $\gamma$ be a geodesic on $M$ with $\gamma(0) = p \in M$ and $V_p, W_p \in T_pM$. Then there exists a unique Jacobi field $Y$ along $\gamma$ such that*

$$Y(0) = V_p \text{ and } Y'(0) = W_p.$$

*Proof.*
The Jacobi field equation (1.2) is a linear system of ordinary differential equations with respect to $Y$. The conditions given above are nothing more than a set of initial conditions for this system which provide a unique solution. □

## 1.2 Tensor Fields

**Definition 1.11.** "Tensor"
*For $r, s \in \mathbb{N}$ and a module $\tilde{V}$ over a ring $\mathbb{K}$ a multilinear function*

$$A : (\tilde{V}^*)^r \times \tilde{V}^s \mapsto \mathbb{K}$$

*is called a r-s-**tensor** over $\tilde{V}$.*

- *$T_s^r$ denotes the set of all r-s-tensors over $\tilde{V}$ and is a module over $\tilde{V}$.*
- *A 0-0-tensor over $\tilde{V}$ is just an element in $\mathbb{K}$.*

**Definition 1.12.** "Tensor Field on a Manifold"
*A r-s-**tensor field** on a manifold $M$ is a r-s-tensor over the $C^\infty$-module $\Gamma(TM)$.*

This means that a $r$-$s$-tensor field $A$ is a $C^\infty$-multilinear function

$$A : (\Gamma(TM^*))^r \times (\Gamma(TM))^s \mapsto C^\infty(M)$$

and $A\left(\overbrace{(\varphi_1, ..., \varphi_r)}^{contravariant\ slot}, \overbrace{(V_1, ..., V_s)}^{covariant\ slot}\right)$ is a function in $C^\infty(M)$

for any set $\{\varphi_1, ..., \varphi_r\} \subset \Gamma(TM)^*$ and $\{V_1, ..., V_s\} \subset \Gamma(TM)$.
From now on we denote for $r, s \in \mathbb{N}$ the set of all $r$-$s$-tensor fields over $M$ by $T_s^r(M)$ and of course a 0-0-tensor field is just a $C^\infty$-function on $M$. Furthermore one can show that $T_s^r$ is a module over $C^\infty(M)$. This definition is very elegant but does not reveal all aspects of tensor fields at first glance. Indeed a tensor field can and should be interpreted as a field in terms of being a map that assigns a tensor over $T_pM$ to every point $p$ on a manifold $M$ in a smooth way. A bit more rigorously explained:
If $A$ is a 0-1-tensor on a manifold $M$, then $A$ assigns $A_p$ to the point $p \in M$, where $A_p$ is a 0-1-tensor on $T_pM$.
Now consider a vector $V_p \in T_pM$: What is the value of $A_p(V_p)$?
The most obvious idea would be the definition

$$A_p(V_p) := (A(W))(p),$$

where $W \in \Gamma(TM)$ is any vector field such that $W(p) = V_p$.
The next lemma tells us, that this definition does not depend on the chosen vector field $W$.

**Lemma 1.4.**
*Let $p \in M$ and $A \in T^r_s(M)$,*
*$\tilde{\varphi}_1, ..., \tilde{\varphi}_r$ and $\varphi_1, ..., \varphi_r$ be one-forms on $M$ such that $\tilde{\varphi}_i \mid_p = \varphi_i \mid_p$,*
*$\tilde{V}_1, ..., \tilde{V}_s$ and $V_1, ..., V_s$ be vector fields on $M$ such that $\tilde{V}_i \mid_p = V_i \mid_p$.*
*then*

$$A(\tilde{\varphi}_1, ..., \tilde{\varphi}_r)(\tilde{V}_1, ..., \tilde{V}_s) = A(\varphi_1, ..., \varphi_r)(V_1, ..., V_s),$$

*Proof.*
A proof can be found in [13, p. 38]. □

**Definition 1.13.** "Pullback"
*Let $M$ and $N$ be two semi-Riemannian manifolds, $\phi : M \mapsto N$ a smooth map between them and $A \in T^0_s(N)$.*
*$\phi^* A$ is called the* ***pullback*** *of $A$ with $\phi$ and is defined by*

$$\phi^* A(V_1, ..., V_s) := A(D\phi(V_1), ..., D\phi(V_s)).$$

$\{V_1, ..., V_s\} \subset \Gamma(TM)$.

**Definition 1.14.** "Tensor Product"
*For $A \in T^r_s(M)$, $B \in T^\alpha_\beta(M)$ one defines:*
$A \otimes B : (\Gamma(TM^*))^{r+\alpha} \times (\Gamma(TM))^{s+\beta} \mapsto C^\infty(M)$,

$$A \otimes B(\varphi_1, ..., \varphi_{r+\alpha}, V_1, ..., V_{s+\beta}) := \\ A(\varphi_1, ..., \varphi_r, V_1, ..., V_s) B(\varphi_1, ..., \varphi_\alpha, V_1, ..., V_\beta).$$

$\{\varphi_1, ..., \varphi_{r+\alpha}\} \subset \Gamma(TM)^*$ *and* $\{V_1, ..., V_{s+\beta}\} \subset \Gamma(TM)$.

The tensor product is obviously $C^\infty(M)$-linear and the tensor product of two $C^\infty$-functions is just the point wise product of these two functions. A convenient aspect of the tensor product is that it provides a tool for expressing a tensor in terms of one-forms and vector fields with respect to a coordinate chart.

**Lemma 1.5.** "Tensor Components w.r.t. a Coordinate Chart"
*Let $X = (x_1, ..., x_n)$ be a coordinate chart on $M$ and $A \in T^r_s(M)$, then*

$$A = \sum_{j_1, ..., j_s, i_1, ..., i_r = 1}^{n} A^{i_1 \dots i_r}_{j_1 \dots j_s} \delta_{i_1} \otimes ... \otimes \delta_{i_r} \otimes dx^{j_1} \otimes ... \otimes dx^{j_s}$$

*with* $A^{i_1...i_r}_{j_1...j_s} := A(dx^1, ..., dx^r, \delta_1, ..., \delta_s)$.

This lemma tells us that a tensor field is determined by the $C^\infty$-functions $A^{i_1...i_r}_{j_1...j_s}$ and that its values on a manifold can be calculated from these functions. Furthermore $\left\{\delta_{i_1} \otimes ... \otimes \delta_{i_r} \otimes dx^{j_1} \otimes ... \otimes dx^{j_s}\right\}$ with $\{i_1, ..., i_r\}$, $\{j_1, ..., j_s\}$ being permutations of $\{1, ..., r\}$ and $\{1, ..., s\}$ form a basis of $T^r_s(M)$.

*Proof.*
The proof is straight forward and can be done by evaluating the tensor field on all appropriate combinations of basis vectors $\delta_i$ and $dx^j$. □

**Lemma 1.6.** "$\alpha$-$\beta$-Contraction"
*For every* $\alpha$ *and* $\beta$ *with* $1 \leq \alpha \leq r$ *and* $1 \leq \beta \leq s$, *there is a unique* $C^\infty$*-linear function on* $T^r_s(M)$

$$C^\alpha_\beta : T^r_s \mapsto T^{r-1}_{s-1},$$

*such that for one-forms* $\varphi_1, ..., \varphi_{r-1}$ *and vector fields* $V_1, ..., V_{s-1}$ *on* $M$

$$\begin{aligned} &C^1_1(\varphi, V) = \varphi(V),\ \varphi \in \Gamma(TM^*),\ V \in \Gamma(TM), \\ &C^\alpha_\beta(A(\varphi_1, ..., \varphi_{r-1}, V_1, ..., V_{s-1})) \\ &= C^1_1(A(\varphi_1, ..., \underbrace{(\cdot)}_{slot\ \alpha}, ..., \varphi_{r-1}, V_1, ..., \underbrace{(\cdot)}_{slot\ \beta}, ..., V_{s-1})). \end{aligned}$$

*For any coordinate chart* $(x^1, ..., x^n)$ *on* $M$ *one gets:*

$$C^\alpha_\beta(A) := \sum_{m=1}^{n} A^{i_1 ... \overbrace{m}^{slot\ \alpha} ... i_r}_{j_1 ... \underbrace{m}_{slot\ \beta} ... j_s} dx^{i_1} \otimes ... \otimes dx^{i_r} \otimes \delta_{j_1} \otimes ... \otimes \delta_{j_s}.$$

*Proof.*
A proof can be found in [13, p. 40]. □

**Remark 1.4.**
*One should be aware of the following fact:*
*For* $A \in T^r_s(M)$, $\varphi \in \Gamma(TM^*)$ *and* $\xi \in \Gamma(TM)$

$$A(\varphi, \xi) = C^1_1(C^1_1(A \otimes \varphi \otimes \xi)).$$

**Remark 1.5.** "Raising and Lowering Indices"
*For $A \in T^r_s(M)$ with components $A^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ in a coordinate chart one defines*

$$A^{i_1 \dots i_{k-1} \ \ i_{k+1} \dots i_{r-1}}_{j_1 \dots j_{k-1} \ i_k \ j_k \dots j_s} := \sum_m g_{i_k m} A^{i_1 \dots i_{k-1} \ m \ i_{k+1} \dots i_r}_{j_1 \dots j_s},$$

$$A^{i_1 \dots i_{k-1} \ j_k \ i_k \dots i_{r-1}}_{j_1 \dots j_{k-1} \ j_{k+1} \dots j_s} := \sum_m g^{j_k m} A^{i_1 \dots i_r}_{j_1 \dots j_{k-1} \ m \ j_{k+1} \dots j_s}.$$

*Here $g_{ij} := g(\delta_i, \delta_j)$ are the components of the metric and $g^{ij}$ the components of the inverse of the matrix $g_{ij}$.*

In the next step we will define a derivative for tensor fields. The idea is to be able to study the rate of change of a tensor field in direction of a tangent vector.

**Definition 1.15.** "Tensor Derivation"
*A **tensor derivation** on a manifold $M$ is a set of $\mathbb{R}$-linear functions*

$$D = \{D^r_s : T^r_s(M) \mapsto T^r_s(M) | r, s \geq 0\}$$

*with the following properties: For $A, B \in T^r_s(M)$ and $D^r_s \in D$:*

1. $D^r_s(A \otimes B) = (D^r_s A) \otimes B + A \otimes (D^r_s B)$,
2. $D^r_s(C^\alpha_\beta(A)) = C^\alpha_\beta(D^r_s A)$ *for any contraction* $C^\alpha_\beta$ *with* $1 \leq \alpha \leq r$ *and* $1 \leq \beta \leq s$.

*$D^r_s$ is then called a tensor derivative.*

**Theorem 1.1.** "Product Rule for Tensor Derivatives"
*For a tensor derivative $D$ and a r-s-tensor field $A \in T^r_s(M)$*

$$\begin{aligned} D\left(A(\varphi_1, ..., \varphi_r, V_1, ..., V_s)\right) = & (DA)(\varphi_1, ..., \varphi_r, V_1, ..., V_s) \\ & + \sum_{i=1}^{r} A(\varphi_1, ..., D\varphi_i, ..., \varphi_r, V_1, ..., V_s) \\ & + \sum_{j=1}^{s} A(\varphi_1, ..., \varphi_r, V_1, ..., DV_j, ..., V_s). \end{aligned}$$

*The left hand side is the derivative of a function while the right hand side only involves derivations of Tensors!*

*Proof.*
A proof for the case $r = s = 1$ can be found in [13, p. 44]. The approach for this case can be applied to any tensor field of higher order. □

Applying the product rule to a one-form $\varphi$ yields:

$$(D\varphi)(V) = D(\varphi(V)) - \varphi(D(V)) \ \textit{for all } V \in \Gamma(TM).$$

Thus this theorem implies that we know how a derivative acts on any tensor field as soon as we know how it acts on $C^\infty$-functions (namely $\varphi(V)$) and vector fields (namely $V$). We will formulate this as a corollary.

**Corollary 1.1.**
*If two tensor derivations $D_1$ and $D_2$ agree on $C^\infty(M)$ and $\Gamma(TM)$, then*

$$D_1 = D_2.$$

This corollary can be used to determine a unique tensor derivation.

**Theorem 1.2.**
*Let $V \in \Gamma(TM)$ be a vector field on $M$ and $\nabla : \Gamma(TM) \mapsto \Gamma(TM)$ a $\mathbb{R}$-linear map that fulfils the Leibniz rule:*

$$\nabla(fW) = (Vf)W + f\nabla(W), \ f \in C^\infty(M), \ W \in \Gamma(TM).$$

*Then there exists a unique tensor derivation $D$ such that*

$$D^0_1 = \nabla \text{ and } D^0_0 = V.$$

*The map $\nabla$ should be regarded as a connection or derivative such as the L.-C. connection or the Lie derivative which will be introduced later.*

*Proof.*
A proof can be found in [13, p. 44]. Again it is only carried out for the case $r = s = 1$ since this suffices to understand the approach. □

**Definition 1.16.** "Covariant Tensor Derivative"
*The covariant derivative of $A \in T^r_s(M)$ with respect to the L.-C. connection $\nabla$ is the r-(s+1)-tensor field **DA** such that*

$$\begin{aligned} DA_\xi(\varphi_1, ..., \varphi_r, V_1, ..., V_s) =& \xi\left(A(\varphi_1, ..., \varphi_r, V_1, ..., V_s)\right) \\ &- A(D_\xi\varphi_1, ..., \varphi_r, V_1, ..., V_s) \\ &- ... - A(\varphi_1, ..., \varphi_r, V_1, ..., \nabla_\xi V_s). \end{aligned}$$

*$\{\varphi_1, ..., \varphi_r\} \subset \Gamma(TM^*)$, $\{V_1, ..., V_s, \xi\} \subset \Gamma(TM)$. The smooth functions $A^{i_1 \cdots i_r}_{j_1 \cdots j_s;k}$ are the components of $DA$ with respect to a coordinate chart.*

**Example 1.1.** "Some Tensors in Semi-Riemannian Geometry"

1. ***The metric***
   *As already mentioned a semi-Riemannian metric is a 0-2-tensor field. It has two contravariant slots and is symmetric and non-degenerate.*

2. ***The Riemannian Curvature Tensor***
   *The Riemannian curvature tensor $R \in T^0_3(M)$ is defined as*

   $$R_{VW}Z := \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z$$

   *for $V, W, Z \in \Gamma(TM)$.*
   *This tensor with components $R_{ijk}$ has three contravariant slots. It measures how the curvature influences the difference between second derivatives. The Term $\nabla_{[V,W]} Z$ is subtracted to make sure it only measures effects of the curvature and not those coming from the Lie bracket of the vector fields involved. In flat spaces $R_{ijk}$ vanishes. Related to $R_{ijk}$ one also defines the tensor $R^i_{jkl}$ by*

   $$R_{ijk} = \sum_l R^l_{ijk} \delta_l$$

   *and one can show that*

   $$R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} - \sum_m \Gamma^i_{lm} \Gamma^m_{kj} + \sum_m \Gamma^i_{km} \Gamma^m_{lj}.$$

   *The covariant slot can be shifted down and one obtains:*

   $$R_{abcd} := g_{an} R^n_{bcd}.$$

   *This is the most common form of the Riemannian curvature tensor.*

3. ***The Ricci Tensor and the Ricci Scalar***
   *The **Ricci tensor** is defined by the contraction*

   $$R_{ab} := \sum_i R^i_{aib}.$$

   *The **Ricci scalar** is the trace of the Ricci tensor:*

   $$R := \sum_{i,j} g^{ij} R_{ij}.$$

4. ***The Volume Element***
*An **orthonormal frame** on $M$ is a set of vector fields $\{e_1, ..., e_n\} \subset \Gamma(TM)$ such that $g(e_i, e_j) = 0$ for $i \neq j$ and $g(e_i, e_i) = \epsilon_i$ for $i, j \in \{1, ..., n\}$ (with $\epsilon_i$ from Definition 1.1). Orthonormal frame fields exist at least locally.*
*A **volume element** on $M$ is a 0-n-tensor field such that*

   *(a) $\omega$ is skew-symmetric,*

   *(b) $\omega(e_1, ..., e_n) = \pm 1$ on an orthonormal frame field.*

*Volume elements always exist at least locally.*

## 1.3 Lie Groups and Lie Algebras

**Definition 1.17.** "Lie Group"
*A **Lie group** $G$ is a differentiable manifold that is also a group with the following properties: For all $a, b \in G$:*

1. $\cdot\,(.,.) : G \times G \mapsto G, (a, b) \mapsto a \cdot b$ *is smooth,*

2. $(.)^{-1} : G \mapsto G, a \mapsto a^{-1}$ *is smooth.*

*where $e \in G$ always denotes the neutral element and $a^{-1} \in G$ is the inverse of $a \in G$.*

**Definition 1.18.** "Lie Subgroup"
*$F$ is a **Lie subgroup** of $G$ if $F$ is a submanifold of $G$ and a subgroup of $G$ at the same time.*
*$F$ is **closed** if it is a closed set in $G$.*

**Definition 1.19.** "Lie Algebra"
*A **Lie algebra** $\tilde{g}$ is a $\mathbb{R}$-vector space equipped with a bracket operator:*
$[.,.] : \tilde{g} \times \tilde{g} \mapsto \tilde{g}$ *such that for all $V, W, Z \in \tilde{g}$*

1. $[.,.]$ *is bilinear,*

2. $[V, W] = -[W, V]$ *(skew-symmetry),*

3. $[[V, W], Z] + [[W, Z], V] + [[Z, V], W] = 0$ *(Jacobi identity).*

*A Lie algebra is called* ***abelian*** *if the bracket operator is zero.*
*A subspace of* $\tilde{g}$ *that is a Lie algebra itself is called a* ***subalgebra****. If not mentioned otherwise* $\tilde{g}$ *will always denote a* $n$*-dimensional Lie algebra.*

**Definition 1.20.** "Structure Constant"
*Let* $\{e_1, ..., e_n\}$ *be a basis of* $\tilde{g}$*. Then one defines the* ***structure constants*** $C^i_{ab}$ *of* $\tilde{g}$ *by:*

$$[e_a, e_b] = \sum_{i=1}^{n} C^i_{ab} e_i.$$

The structure constants fulfil

1. $C^i_{ab} = -C^i_{ba}$ for $i, a, b \in \{1, ..., n\}$,
2. $\sum_m C^l_{im} C^m_{jk} + C^l_{jm} C^m_{ki} + C^l_{km} C^m_{ij} = 0.$

Point 1 accounts for the skew-symmetry of the bracket and point 2 for the Jacobi identity in Definition 1.19.

**Definition 1.21.** "Derived Algebra"
*The* ***derived algebra*** $D\tilde{g}$ *of* $\tilde{g}$ *is the Lie algebra that arises from the vectors*

$$\{\ [e_i, e_j] \mid i, j \in \{1, ..., n\}\ \}$$

*and* $D^k\tilde{g}$ *is the derived algebra of* $D^{k-1}\tilde{g}$*.*
*Furthermore one defines:*

$$C^1\tilde{g} := [\tilde{g}, \tilde{g}],\ C^k\tilde{g} := [\tilde{g}, C^{k-1}\tilde{g}].$$

- $\tilde{g}$ *is* ***solvable*** *(or integrable) if* $D^k\tilde{g} = \{0\}$ *for some* $k \in \mathbb{R}$*.*
- $\tilde{g}$ *is* ***nilpotent*** *if for some* $k \in \mathbb{R}$ $C^k\tilde{g} = \{0\}$*.*

Of course every nilpotent Lie algebra is solvable.

**Definition 1.22.** "Solvable Radical"
*The largest solvable ideal of* $\tilde{g}$ *(an ideal is a subalgebra* $R$ *such that* $[R, \tilde{g}] \subseteq R$*) is unique and is called the* ***solvable radical*** *of* $\tilde{g}$*.*

**Definition 1.23.** "Simple/Semi-simple Algebra"

- *$\tilde{g}$ is **simple** if it is neither abelian nor solvable.*
- *$\tilde{g}$ is **semi-simple** if its solvable radical $R$ is zero.*
  *This is equivalent to the fact that $\tilde{g}$ is the sum of simple algebras.*

**Remark 1.6.**
*The theorem about the **Levi decomposition** states that any finite dimensional real Lie algebra is (as a vector space) the direct sum of a solvable subalgebra $R$ and a semi-simple subalgebra $S$. One has $[S,S]=S,\ [R,R]\subset R$ and $[R,S]\subseteq R$. This is called a **semi-simple product**. It is a direct product if $[S,R]=0$. A proof of this theorem can be found in [12, pp. 3-17].*

**Definition 1.24.** "Left-Multiplication"
*On a group $(G,\ \cdot)$ one defines the left-multiplication as*

$$l_a : G \mapsto G,\ b \mapsto l_a(b) := a \cdot b,\ for\ a, b \in G.$$

**Definition 1.25.** "Left-invariant Vector Field"
*Let $G$ be Lie group.*
*$V \in \Gamma(TG)$ is called a **left-invariant vector field** if*

$$Dl_a(V) = V \text{ for all } a \in G.$$

This yields $Dl_a(V_b) = V_{a \cdot b}$ and that left-invariant vector fields are smooth. It can be shown that every left-invariant vector field is complete.

**Definition 1.26.** "Lie Algebra of a Lie Group"
*The vector space $\tilde{g}$ of all left-invariant vector fields of a Lie group $G$ equipped with the Lie bracket (see Definition 1.3) is a Lie algebra usually referred to as the **Lie algebra of the Lie group** $G$.*

**Lemma 1.7.**
*$T_eG$ is isomorphic to $\tilde{g}$ due to the isomorphism that maps $V \in \tilde{g}$ to $V_e$.*

*Proof.*
It is clear that the map $\tilde{g} \mapsto T_eG,\ V \mapsto V_e$ is linear and surjective. It is injective since $V_e = 0$ yields $V_a = Dl_a(V_e) = 0$ for $a \in G$. □

**Remark 1.7.** "Lie Algebras Generate Lie Groups"
*Every Lie algebra defines a unique, simply-connected Lie Group.*
*Therefore a basis of the Lie algebra is called the* ***generator*** *of this Lie group. The term simply-connected is crucial here: We will see in Example 1.3 that two or more Lie groups can have the same Lie algebra. At least all connected Lie groups that are generated by the same Lie algebra are* ***homomorphic****.*

*Proof.*
This theorem and a proof for it can be found in [4, § 8.1]. □

**Definition 1.27.** "One-Parameter Subgroup"
*A* ***one-parameter subgroup*** *in a Lie group $G$ is a smooth homomorphism*

$$\gamma : (\mathbb{R}, +) \mapsto G$$

*such that* $\gamma(s+t) = \gamma(s) \cdot \gamma(t)$, $\gamma(0) = e$, $\gamma(-t) = \gamma(t)^{-1}$.

**Lemma 1.8.**
*Every one-parameter subgroup of a Lie group $G$ corresponds to exactly one maximal integral curve of a left-invariant vector field in $\tilde{g} \cong T_eG$.*

*Proof.*
First we prove: *$\gamma$ is a one-parameter subgroup implies $\gamma$ is the maximal integral curve of a left-invariant vector field.*
Let $\gamma : \mathbb{R} \mapsto G$ be a one-parameter subgroup. Then it can be interpreted as a smooth curve on $G$. The vector field $\gamma'$ is left-invariant since for $s \in \mathbb{R}$:

$$\begin{aligned} Dl_{\gamma(s)}(\gamma'(0)) &:= \lim_{t \to 0} \frac{(l_{\gamma(s)} \circ \gamma(t)) - (l_{\gamma(s)} \circ \gamma(0))}{t} \\ &= \lim_{t \to 0} \frac{(\gamma(s) \cdot \gamma(t)) - \gamma(s)}{t} \\ &= \lim_{t \to 0} \frac{\gamma(s+t) - \gamma(s)}{t} =: \gamma'(s). \end{aligned}$$

Thus the one-parameter subgroup $\gamma$ corresponds to the unique left-invariant vector field represented by $\gamma'(0)$ and of course $\gamma$ is the integral curve of $\gamma'$.

Now we prove: *$\gamma$ is the maximal integral curve of a left-invariant vector field* implies *$\gamma$ is a one-parameter subgroup.*

Let $V \in \tilde{g}$ be a left-invariant vector field and $\gamma : [a, b] \mapsto G,\ \gamma(0) = e$ be its maximal integral curve. Then $\gamma$ fulfils the properties of a one-parameter subgroup:

1. $\gamma$ is defined on the whole real line since left-invariant vector fields are complete.
2. The integral curve of $V$ at the point $\gamma(s)$ is the map $t \mapsto \gamma(s + t)$. Due to the left-invariance of $V$ one also knows that the integral curve through $\gamma(s)$ is the map t $\mapsto \gamma(s) \cdot \gamma(t)$. This yields $\gamma(s+t)=\gamma(s) \cdot \gamma(t)$ since local integral curves are unique.

□

**Definition 1.28.** "Lie Exponential Map"
*Let $G$ be a Lie group and $\tilde{g}$ its Lie algebra.*

$$exp : \tilde{g} \mapsto G,\ V \mapsto \gamma_V(1)$$

*is called the* ***Lie exponential map****, where $\gamma_V$ is the one-parameter subgroup corresponding to $V \in \tilde{g} \approx T_eG$.*

The Lie exponential map works just like its analog on smooth manifolds, mapping a neighbourhood of $0 \in \tilde{g}$ diffeomorphically to a neighbourhood of $e \in G$.

**Example 1.2.** "$GL(n, \mathbb{R})$ as a Lie Group"
*The set of all $n \times n$ matrices over $\mathbb{R}$, $gl(n, \mathbb{R})$, is a vector space and with respect to the bracket operator $[A, B] := AB - BA$ for $A, B \in gl(n, \mathbb{R})$ it forms a group. Thus it is a Lie algebra. The set of all invertible matrices, $GL(n, \mathbb{R})$, is a manifold as well as a group with respect to matrix multiplication. Therefore it is a Lie group. In fact $gl(n, \mathbb{R})$ is isomorphic to the Lie algebra of $GL(n, \mathbb{R})$.*
*The Lie exponential map is defined as*
$exp : gl(n, \mathbb{R}) \mapsto GL(n, \mathbb{R}),\ V \mapsto exp(V),$

$$exp(V) := \sum_{n=0}^{\infty} \frac{1}{n!} V^n,\ for\ V \in gl(n, \mathbb{R}).$$

*For instance by defining the subgroup* $\gamma : \mathbb{R} \mapsto GL(n, \mathbb{R})$,

$$\gamma(t) := \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & \mathrm{e}^t \end{pmatrix}$$

*one gets*

$$\gamma(t) = exp(t \cdot V), \ \textit{with } V = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Example 1.3.** "$o(n)$ as the Lie Algebra of $O(n)$"

- $O(n) := \{M \in gl(n, \mathbb{R}^n) \mid M^T = M^{-1}\}$ *(orthogonal matrices).*
- $o(n) := \{M \in gl(n, \mathbb{R}^n) \mid M^T = -M\}$ *(skew-symmetric matrices).*

*$O(n)$ is a group with respect to matrix multiplication (id denotes the identity matrix which is the neutral element of $O(n)$). It is a manifold because it is the null set of the map* $h : gl(n, \mathbb{R}^n) \mapsto gl(n, \mathbb{R}^n)$,

$$h(A) := (AA^T) - id.$$

*Let $T_{id}(O(n))$ denote the Lie algebra of $O(n)$. Then*

$$o(n) \textit{ is isomorphic to } T_{id}(O(n))$$

*and therefore often regarded as the Lie algebra of $O(n)$.*

*Proof.*
Let $A(t)$ be a curve on $O(n)$ such that $A(0) = id$.

$$\begin{aligned} A(t) \in O(n) &\Rightarrow A(t)A(t)^T = id \\ &\Rightarrow \frac{d}{dt}\left((A(t)A(t)^T\right) = 0 \\ &\Rightarrow \left(\frac{d}{dt}A(t)\right) A(t)^T = A(t)\frac{d}{dt}\left(A(t)\right)^T \\ &\Rightarrow \frac{d}{dt}|_0 A(t) = \left(\frac{d}{dt}|_0 A(t)\right)^T. \end{aligned}$$

Since every element in $T_{id}(O(n))$ is of the form $\frac{d}{dt}|_0 A(t)$ for some curve $A(t)$ in $O(n)$ this proves the claim. □

*$O(n)$ has two disjoint components. The component $O^+(n)$ contains the matrices that have positive determinant and is often referred to as $SO(n)$. The other component, $O^-(n)$, contains the matrices with negative determinant. $O^+(n)$ contains the identity map and is a Lie subgroup of $O(n)$. Thus $o(n)$ is also the Lie algebra of $O^+(n) = SO(n)$ and therefore often referred to as $so(n)$.*

**Remark 1.8.** "Semi-Orthogonal Group $O_m(n)$"
*Let $O_m(n)$ denote the set of all matrices that preserve the scalar product $g$ on $\mathbb{R}^n_m$. Then $O_m(n)$ can be identified with the set of all isometric isomorphisms on $\mathbb{R}^n_m$.*
*$O_m(n)$ is a closed subgroup of $GL(n, \mathbb{R})$ and therefore a Lie group itself, called the **semi-orthogonal group**. The following equivalence holds:*

$$\Leftrightarrow 1)\ A \in O_m(n),$$
$$\Leftrightarrow 2)\ A^t = SA^{-1}S \ with\ S\ defined\ as\ in\ Remark\ 1.1,$$
$$\Leftrightarrow 3)\ A\ maps\ an\ orthonormal\ basis\ to\ an\ orthonormal\ basis.$$

*Proof.*
A proof of these equivalences can be found in [13, p. 234]. □

**Example 1.4.** "Lie Algebra of $O_m(n)$"

1. *Define $o_m(n) := \{A \in gl(n, \mathbb{R}^n_m) \mid A^T = S^{-1}AS = -SAS\}$ with $S \in gl(n, \mathbb{R})$ from Definition 1.1. Then*

   $$o_m(n) \text{ is the Lie algebra of } O_m(n).$$

2. *The elements $A$ in $o_m(n)$ are of the form*

   $$\begin{pmatrix} a & x \\ x^T & b \end{pmatrix},$$

   *where $a \in o(m)$, $b \in o(n-m)$ and $x$ is a $n \times m$ matrix.*

3. *$dim(O_m(n)) = dim(o_m(n)) = \frac{n(n-1)}{2}$.*

*The first claim allows us to regard $o_m(n)$ as the skew-adjoint operators in $\mathbb{R}^n_m$.*

*Proof.*
The proof of this claim can be found in [13, p. 235]. □

**Remark 1.9.** "Differences between $O(n)$ and $O_m(n)$"

- $O(n)$ *is* ***compact*** *and has* ***two*** *disjoint components:* $O^+(n)$ *and* $O^-(n)$.
- $O_m(n)$ *is* ***closed*** *but* ***unbounded*** *and has* ***four*** *disjoint, open components.*

*Proof.*
A proof for these statements can be found in [13, pp. 236-237]. □

## 1.4 Basics in General Relativity Theory

This section follows the lecture notes by S. M. Carrol [2] and their introductory abstract [3]. The basic idea in general relativity is that space and time are merged together into **spacetime**, which is represented by a time oriented, 4-dimensional Lorentzian manifold with a metric of signature $(- + ++)$ or$(+ - --)$ (a definition of the term time oriented is given below). This metric has to obey the **Einstein equation**:

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}.$$

The left hand side only involves curvature related terms. The right hand side involves $G$ as Newton's constant of gravitation and the **energy-momentum tensor** $T_{ab}$ containing all necessary information on energy and momentum of a matter field. Thus this equation relates curvature of spacetime to matter and its properties. Roughly speaking, this is often referred to by saying "matter curves spacetime". It also yields that gravity is no more a force as it is in Newtonian theory. The tangent spaces of a spacetime manifold look like Minkowski space, i.e. $\mathbb{R}^4$ equipped with the Lorentzian scalar product:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

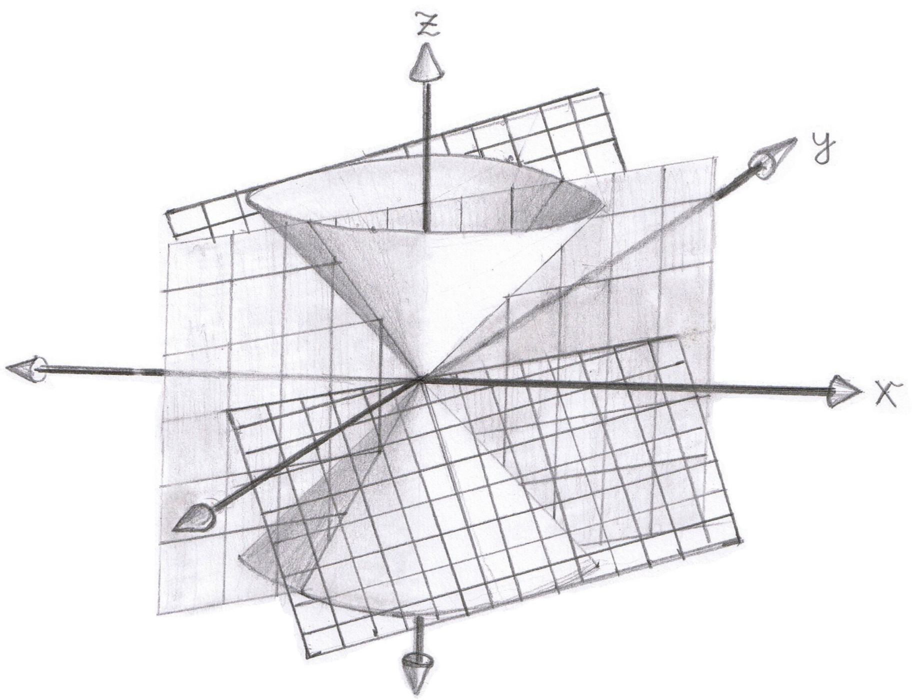
A Lorentzian metric on a manifold allows us to distinguish three different types of tangent vectors $V \in T_pM$ or trajectories (according to the type of their tangent vectors) by their **causal character**:

1. time-like: $g(V, V) < 0$,
2. light-like: $g(V, V) = 0$,
3. space-like: $g(V, V) > 0$.

This terminology accounts for the following fact: In general relativity theory freely falling test bodies always move along a geodesic. In the case where the body is massless this geodesic will have tangent vectors with vanishing length: $g(V, V) = 0$. In the case of a massive body $g(V, V)$ will be negative. In 3-dimensional Minkowski space all light-like tangent vectors are tangential to a double cone as shown on the picture below. This is why the cone is referred to as the **light-cone**. Inside this cone tangent vectors are all time-like and tangent vectors laying outside the cone are space-like.
The same classification can be made for a tangent subspace of dimension greater or equal than two: On a Lorentzian manifold $M$ let $W$ be such a subspace of $T_pM$.

1. If $g|_W$ is positive definite, $W$ is **space-like**.
2. If $g|_W$ is non-degenerate and of signature $(1, n-1)$, $W$ is **time-like**.
3. If $g|_W$ is degenerate, $W$ is **light-like**.

The next picture shows what the different types of planes and the lightcone in 3-dimensional Minkowski space look like.




**Remark 1.10.**
*Sometimes the metric for a spacetime is given with signature $(+ - --)$. In this case the terms light-like and space-like have to be switched in the definitions given above. In Chapter 4 the metrics have signature $(+ - --)$ in order to stick as close as possible to the notation in the papers they have been published in. Furthermore, if not mentioned otherwise, we will always use coordinates $(x_1, x_2, x_3, x_4) = (t, x, y, z)$ in a spacetime.*

The term **time oriented** mentioned above means that there exists a unit vector field on $M$ that is time-like at every point. Such a vector field is called an **observer field**. The covariant derivative of an observer field $V$ can be decomposed as follows:

$$V_{i;k} = \omega_{ik} + \sigma_{ik} + \frac{1}{N}\theta P_{ik} - \dot{V}_i V_k.$$

(See [17]).

- $\omega_{ik}$ is the antisymmetric part, called **rotation**.
- $\sigma_{ik}$ is the symmetric, traceless part, called the **shear**.
- $\theta$ is the trace also called **expansion**.
- $P_{ik} = g_{ik} - V_i V_k$ is the projection tensor on the hypersurfaces perpendicular to $V$.
- $\dot{V}_i = \sum_a V^a V_{i;a}$ is called the **acceleration**.
- If $V$ is a conformal Killing vector field it is **parallax-free** (Conformal Killing vector fields will be introduced in the next chapter).

In general it is not easy to find solutions of the Einstein field equations. Simplifying assumptions make it somewhat easier: E. g. assuming a spherically symmetric matter distribution in a vacuum (i.e. $R_{ij} = 0$) yields the **Schwarzschild metric**. This metric can be used to describe the gravitational fields in solar systems as well as planets or black holes. This metric and the mathematical definition of the term spherically symmetric is presented in Chapter 3. In Chapter 4 the Gödel metric will be introduced as a further solution.

# Chapter 2

# Killing Vector Fields

## 2.1 Definitions and Properties

Killing vector fields describe certain symmetries on a manifold since they turn out to be closely related to isometries. Therefore we will start with a definition and some properties of isometries on manifolds. After that we will examine the basic concepts behind Killing vector fields and some important examples.
Mostly we will be dealing with the metric tensor. Therefore we will formulate some theorems only for 0-2-tensors.
In sections 3, 4 and 5 an approach to classify Lie algebras is presented, since Killing vector fields turn out to form a Lie algebra. These classifications are of great importance in general relativity theory, where spacetimes are classified by several criteria, among them Killing vector fields and the resulting physical properties. This aspect is explained in the Chapter 3.

**Definition 2.1.** "Isometry"
*An* ***isometry*** *between two semi-Riemannian manifolds* $(M, g_M)$ *and* $(N, g_N)$ *is a diffeomorphism* $\phi : M \mapsto N$ *such that*

$$g_N\left(D\phi(V_p), D\phi(W_p)\right) = g_M(V_p, W_p)$$

*for all* $V_p, W_p \in TM$.

Obviously the set $I(M)$ of all isometries on $M$ is a group under the composition of mappings.

**Lemma 2.1.**
*Let* $M$ *and* $N$ *be two semi-Riemannian manifolds and* $M$ *be connected. If*

*two isometries $\phi, \psi : M \to N$ fulfil $\phi(p) = \psi(p)$ at one point $p \in M$ and $D\phi(V_p) = D\psi(V_p)$ for all $V_p \in T_pM$ then*

$$\phi = \psi.$$

*Proof.*
Assume $\phi, \psi : M \to N$ were both local isometries such that $\phi(p) = \psi(p)$ and $D\phi(V_p) = D\psi(V_p)$ at a point $p \in M$.
Define $A := \{q \in M \mid D_\phi(V_q) = D\psi(V_q) \ for \ all \ V_q \in T_qM\}$.
Obviously $A$ is closed in $M$ and non-empty since $D\phi - D\psi$ is continuous. We will prove that $A$ is also open by showing that for every $q \in A$ the normal neighbourhood $U_q$ is contained in $A$:
A point $r \in U_q$ is contained in the image of the exponential map if $r = exp_q(V_q) = \gamma_{V_q}(1)$ for some $V_q \in T_qM$. Now apply the isometry properties:

$$\phi(r) = \phi(\gamma_{V_q}(1)) = \gamma_{D\phi(V_p)}(1) = \gamma_{D\psi(V_p)}(1) = \psi(\gamma_{V_q}(1)) = \psi(r).$$

The second equality holds since isometries map geodesics to geodesics. This means the unique geodesic through $q$ and $r$ is mapped to the unique geodesic through $\phi(q)$ and $\phi(r)$. The third equality holds by assumption. □

**Example 2.1.** "Isometry Group of a Pseudosphere"
*Let $S^n_m$ be the n-dimensional pseudosphere in $\mathbb{R}^n_m$. Then for $m < n$*

$$I(S^n_m) = O_m(n+1).$$

*Proof.*
The proof basically relies on the fact that $O_m(n+1)$ acts transitively on $S^n_m$. The details can be found in [13, p. 239]. □

**Definition 2.2.** "Lie Derivative of a Vector Field"
*For vector fields $V, W, Z \in \Gamma(TM)$ one defines the Lie derivative as*

$$L_V(f) := Vf \ for \ f \in C^\infty(M),$$
$$L_V(W) := [V, W].$$

*According to Theorem 1.2 $L_V(W)$ defines a tensor derivation since the Leibnitz rule holds:*

$$L_V(fW) = [V, fW] = Vf + f[V, W].$$

**Lemma 2.2.**
*For $V, W \in \Gamma(TM)$, $A \in T_s^0(M)$ and the local flow $\phi$ of $V$ near $p \in M$*

1. $L_V(W)|_p = [V,W]_p = \lim_{t\to 0}\left(\frac{d\phi_{-t}(W_{\phi_t(p)})-W_p}{t}\right)$,
2. $L_V(A)|_p = \lim_{t\to 0}\left(\frac{\phi_t^*(A)-A_p}{t}\right)$,

*where $\phi_{-t}^*$ is the pullback of a tensor field from Definition 1.13.*

*Proof.*

1. Three cases need to be considered:
   1) $V = 0$ on a neighbourhood of $p$.
   2) $V \neq 0$ on a neighbourhood of $p$.
   3) $V_p = 0$ but $V_{p_i} \neq 0$ where $p_i$ is a sequence on $M$ with $\lim_{i\to\infty} p_i = p$.

   Case 1)
   In this case one has: $\phi_t = id$.
   Therefore the equation holds since the limit vanishes.

   Case 2)
   We choose a coordinate system $x^i$ on a sufficiently small neighbourhood $U_p$ of $p$ such that $V = \delta_1$. This yields:

   $$\begin{aligned}
   & x^1(\phi_t(q)) = x^1(q) + t\ ,\ x^i(\phi_t(q)) = x^i(q)\ for\ j \neq 1,\ q \in U_p \\
   & \Rightarrow\ D\phi_t(\delta_i) = \delta_i\ for\ all\ i \\
   & \Rightarrow\ D\phi_{-t}(W_{\phi_t(p)}) = \sum_i W^i(\phi_t(p))\delta_i|_p \\
   & \Rightarrow\ \lim_{t\to 0}\left(\frac{D\phi_{-t}(W_{\phi_t(p)}) - W_p}{t}\right) = \sum_i \left(\frac{d}{dt}|_0\left(W^i \circ \phi_t(p)\right)\right)\delta_i|_p \\
   & \qquad = \sum_i (\delta_1|_p W^i)\delta_i|_p = [\delta_1, W]_p = [V,W]_p.
   \end{aligned}$$

   Case 3)
   Since both Terms, $[V,W]$ and $d\phi_{-t}(W_{\phi_t(p)})$, depend continuously on $p$ one can apply case 2) to the points $p_i$ and then transfer the result by taking the limit $i \to \infty$.

2. We will only proof this for a 0-2-tensor field $A$. The product rule for tensor derivatives yields

   $$L_V A(Z,W) = VA(Z,W) - A([V,Z],W) - A(Z,[V,W]). \quad (2.1)$$

Now we transform the right hand side of claim 2:

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\left(\phi_t^*(A)-A_p\right)(Z_p,W_p)\\
&=\lim_{t\to 0}\frac{1}{t}\left(A\left(D\phi_t(Z_p),D\phi_t(W_p)\right)-A\left(Z_p,W_p\right)\right)\\
&=\lim_{t\to 0}\frac{1}{t}\left(A\left(D\phi_t(Z_p),D\phi_t(W_p)\right)-A\left(Z_{\phi_t(p)},W_{\phi_t(p)}\right)\right) &(2.2)\\
&\quad+\lim_{t\to 0}\frac{1}{t}\left(A\left(Z_{\phi_t(p)},W_{\phi_t(p)}\right)-A\left(Z_p,W_p\right)\right). &(2.3)
\end{aligned}$$

Now we have to transform these limits into the terms of the right hand side of equation (2.1). We begin with the limit (2.2):

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\left(A\left(D\phi_t(Z_p),D\phi_t(W_p)\right)-A\left(Z_{\phi_t(p)},W_{\phi_t(p)}\right)\right)\\
&=\lim_{t\to 0}\frac{1}{t}A\left(D\phi_t(Z_p)-Z_{\phi_t(p)},D\phi_t(W_p)\right) &(2.4)\\
&\quad+\lim_{t\to 0}\frac{1}{t}A\left(Z_{\phi_t(p)},D\phi_t(W_p)-W_{\phi_t(p)}\right)\\
&=\lim_{t\to 0}\frac{1}{t}A\left(D\phi_t\left(Z_p-D\phi_{-t}(Z_{\phi_t(p)})\right),D\phi_t\left(W_p\right)\right)\\
&\quad+\lim_{t\to 0}\frac{1}{t}A\left(Z_{\phi_t(p)},D\phi_t\left(W_p-D\phi_{-t}(W_{\phi_t(p)})\right)\right)\\
&=-A\left(D\phi_0\left(\lim_{t\to 0}\frac{1}{t}D\phi_{-t}(Z_{\phi_t(p)})-Z_p\right),\lim_{t\to 0}D\phi_t\left(W_p\right)\right)\\
&\quad-A\left(\lim_{t\to 0}Z_{\phi_t(p)},D\phi_0\left(\lim_{t\to 0}\frac{1}{t}D\phi_{-t}(W_{\phi_t(p)})-W_p\right)\right)\\
&=-A([V,Z]_p,W_p)-A(Z_p,[V,W]_p).
\end{aligned}$$

The last equality uses claim 1.

The limit in (2.3) can be transformed with less effort:
Let $\alpha$ be the integral curve of $V$ through $p$: $\phi_t(p)=\alpha(t)$.

$$\begin{aligned}
&\lim_{t\to 0}(A\left(Z_{\phi_t(p)},W_{\phi_t(p)}\right)-A\left(Z_p,W_p\right)\\
&=\frac{d}{dt}|_0 A\left(Z_\alpha,W_\alpha\right)\\
&=\alpha'(0)A\left(Z,W\right)=V_pA\left(Z,W\right).
\end{aligned}$$

□

**Remark 2.1.** "Lie Derivative w.r.t Coordinates"
*With respect to a coordinate chart the components of the Lie derivative of a 0-2-tensor $g$ are*

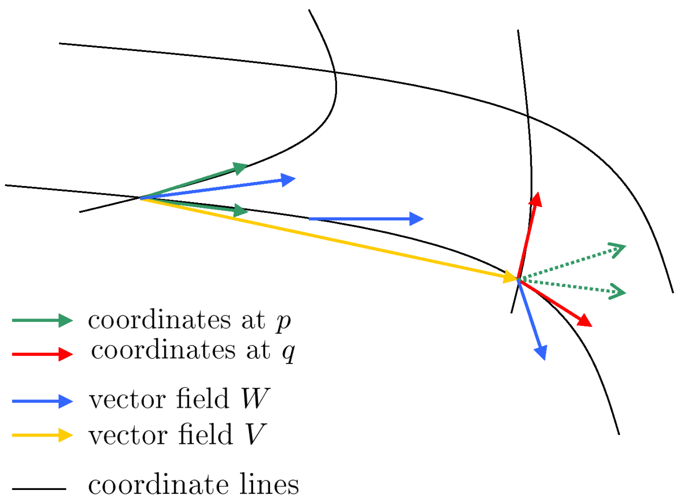
$$(L_V g)_{ij} = \sum_{k=1}^{n} g_{ij,k} V^k + g_{kj} V^k_{,i} + g_{ik} V^k_{,j}.$$

*Proof.*

$$\begin{aligned}(L_V g)(\delta_i, \delta_j) = & \; L_V(g_{ij}) - g(L_V \delta_i, \delta_j) - g(\delta_i, L_V \delta_j) \\ = & \; \sum_a V^a g_{ij,a} - g(e_j, \sum_a V^a \nabla_{\delta_a} \delta_i - V^a_{,i} \delta_a - V^a \nabla_{\delta_i} \delta_a) \\ & - g(\delta_i, \sum_a V^a \nabla_{\delta_a} \delta_j - V^a_{,j} \delta_a - V^a \nabla_{\delta_j} \delta_a) \\ = & \; \sum_{k=1}^{n} g_{ij,k} V^k + g_{kj} V^j_{,i} + g_{ik} V^k_{,j}.\end{aligned}$$

□

One can think of the Lie derivative in the following way:
Let $p$ be a point in $M$ and $q$ its infinitesimal close neighbour such that $p - q = V_p$. Let $W$ be a vector field on $M$. Imagine one goes from $p$ to $q$ following $V_p$ taking the coordinates at $p$ with oneself, carrying them to $q$. Now one measures the vector $W_q$ with respect to the "old" coordinates at $p$ and compares it to the value of $W_p$ with respect to the coordinates at $p$. Together with the sketch below this is a quite pictorial description of what the Lie derivative does.



**Lemma 2.3.**
*If $\omega$ is a local volume element on $M$ and $V \in \Gamma(TM)$ then*

$$L_V(\omega) = div(V)\omega.$$

*Proof.*
Let $\{e_1, ..., e_n\}$ be an orthonormal frame field along the integral curve of $V$ through $p \in M$. This means that $\omega(e_1, ..., e_n) = 1$. Define $[V, e_i]$ as $\sum_k h_{ik} e_k$ with $h_{ik} \in C^\infty(M)$.

$$(L_V\omega)(e_1, ..., e_n) = -\sum_k \omega(e_1, ..., [V, e_k], ..., e_n) = -\sum_k h_{kk}.$$

The last equality holds due to the skew symmetry of $\omega$. One also has

$$\begin{aligned} div(V) &= \sum_k g(\nabla_{e_k} V, e_k) = -\sum_k g([V, e_k], e_k) + \sum_k g(\nabla_V e_k, e_k) \\ &= -\sum_k h_{kk} + \sum_k \frac{1}{2} V g(e_k, e_k) = -\sum_k h_{kk}. \end{aligned}$$

The last equality holds since $g(e_k, e_k)$ is constant. □

**Definition 2.3.** "Killing Vector Field"
*$\xi \in \Gamma(TM)$ is a **Killing vector field** if*

$$L_\xi(g) = 0,$$

*where $g$ is the metric on $M$.*
*$\xi$ is a **conformal Killing vector field** if for some $\lambda \in \mathbb{R}$*

$$L_\xi(g) = \lambda\, g.$$

*(The definition of a conformal Killing vector is only given in order to be able to define a parallax-free observer field in sections 1.4 and 4.3.)*
*The $\mathbb{R}$-**vector space** of all Killing vector fields on a manifold M is denoted by **i(M)**. Lemma 2.6 will prove that $i(M)$ is finite dimensional. The elements of a basis of i(M) are often referred to as the **Killing vectors of** $(M, g)$.*

**Remark 2.2.** "Killing Equation"
*Remark 2.1 yields that the condition $L_\xi g = 0$ for a Killing vector field can be expressed as a linear system of differential equations of first order:*

$$0 = (L_\xi g)_{ij} = \sum_{k=1}^{n} g_{ij,k}\xi^k + g_{kj}\xi^k_{,i} + g_{ik}\xi^k_{,j} \;\Leftrightarrow\; \xi_{i;j} = -\xi_{j;i}.$$

**Remark 2.3.**
*The covariant tensor derivative of the metric with respect to the Levi-Cevita connection vanishes:*

$$\forall\ V \in \Gamma(TM):\ D_V g = 0.$$

*This is clear since*

$$Vg(Z,Y) = g(\nabla_V Z, Y) + g(Z, \nabla_V Y)\ for\ V, Y, Z \in \gamma(TM)$$

*must hold for a L.-C. connection. Lemma 2.2 suggests to interpret the Lie derivative as the rate of change of a vector field $W$ or tensor $A$ under the flow of $V$. This does not necessarily vanish! Therefore one can regard a Killing vector field as an infinitesimal isometry since the metric does not change under its flow. The following lemma confirms this intuition.*

**Lemma 2.4.**
*$\xi \in \Gamma(TM)$ is a Killing vector field if and only if all the stages $\phi_t$ of its local flow are* ***isometries*** *at every point on $p \in M$.*

*Proof.*
Assume that $\phi_t$ is an isometry for every $t \in \mathbb{R}$ that is sufficiently small. Then $\phi_t^*(g) = g$ and therefore according to Lemma 2.2 $L_\xi(g) = 0$.
Now assume $L_\xi(g) = 0$. Since $\phi_s \circ \phi_t = \phi_{s+t}$ Lemma 2.2 yields for sufficiently small $s \in \mathbb{R}$ and $V \in \Gamma(TM)$

$$\lim_{t \to 0} \frac{g(D\phi_{s+t}(V), D\phi_{s+t}(V)) - g(D\phi_s(V), D\phi_s(V))}{t} = 0.$$

This means $g(d\phi_s(V), d\phi_s(V))$ is constant and therefore

$$g(d\phi_s(V), d\phi_s(V)) = g(V, V),$$

which is the defining property of an isometry (see Definition 2.1). $\square$

**Corollary 2.1.**
*Let $\omega$ be the volume element on an orientated manifold $M$ and $\xi$ a Killing vector field on $M$ then:*

$$L_\xi \omega = 0.$$

*Proof.*
The proof is a straight forward calculation of the term $L_\xi \omega$. But it can also be done by sticking to the intuition from Lemma 2.2 of a Killing vector field as an infinitesimal isometry: The volume element takes value 1 on any orthonormal basis of a tangent space. All stages of the flow $\phi$ of a Killing vector field are isometries. Thus an orthonormal basis is mapped to an orthonormal basis under $D\phi$. Therefore $D\phi_t^*(\omega) = \omega$ for every sufficiently small $t \in \mathbb{R}$. □

**Theorem 2.1.** "Killing Vectors and Skew-Adjointness"
*The following equivalence for $\xi \in \Gamma(TM)$ holds:*

$$\begin{aligned} &1)\ \xi \text{ is a Killing vector field} \\ \Leftrightarrow\ &2)\ g(\nabla_Z \xi, Y) = -g(\nabla_Y \xi, Z)\ for\ Z, Y \in \Gamma(TM). \end{aligned}$$

*Point 2) means that $D\xi$ is* ***skew-adjoint*** *with respect to the metric $g$.*

*Proof.*

$$\begin{aligned} &L_\xi g = 0 \\ \Leftrightarrow\ &\xi g(Z, Y) = g([\xi, Z], Y) + g(Z, [\xi, Y]) \\ \Leftrightarrow\ &-g([\xi, Z], Y) + g(\nabla_\xi Z, Y) - g([\xi, Y], Z) + g(\nabla_\xi Y, Z) = 0 \\ \Leftrightarrow\ &g(\nabla_Z \xi, W) + g(\nabla_Y \xi, Z) = 0. \end{aligned}$$

□

**Lemma 2.5.** "Conservation Lemma"
*Let $\xi$ be a Killing vector field on $M$ and $\gamma$ be a geodesic on $M$.*
*Then $\xi_{\gamma(t)}$ is a Jacobi field along $\gamma$ and $g(\gamma', \xi_\gamma)$ is constant along $\gamma$.*

*Proof.*
Since the local flow $\phi_s$ of $\xi$ is an isometry it maps geodesics to geodesics. Therefore locally $h(s,t) := \phi_s(\gamma(t))$ is a geodesic variation and $\xi$ is a Jacobi field, since $\xi_{\gamma(t)} = \frac{\partial}{\partial s}|_{(0,t)} \phi_s(\gamma(t)) = \frac{\partial}{\partial s}|_{(0,t)} h(s,t)$ as explained in Lemma 1.2. $\xi$ is constant along a geodesic since

$$\frac{d}{dt} g(\xi_\gamma, \gamma') = g(\xi'_\gamma, \gamma') = g(\nabla_{\gamma'} \xi, \gamma') = 0.$$

The last equality makes use of the skew-adjointness of $\nabla\xi$ from Theorem 2.1. □

**Lemma 2.6.** "Uniqueness Conditions for a Killing Vector Field"

1. *Let $\xi$ be a Killing vector field on a* ***connected*** *semi-Riemannian manifold $M$ such that $\xi_p = 0$ and $\nabla_{V_p}\xi = 0$ for all $V_p \in T_pM$. Then*

$$\xi = 0.$$

   *This yields that a Killing vector field is uniquely determined by its value $\xi_p$ at a point $p \in M$ and the value of its derivative $\nabla\xi$ at $p$.*

2. *The following relation between a Killing vector field and the curvature tensor holds:*

$$\xi_{k;b;a} = \sum_m R^m_{abk}\xi_m. \tag{2.5}$$

3. *The number of Killing vectors on a connected $n$-dimensional semi-Riemannian manifold is less or equal $\frac{n(n+1)}{2}$ and only spaces with constant curvature (i.e. $R_{abcd} = \frac{R}{n(n-1)}(g_{bd}g_{ac} - g_{bc}g_{ad})$, $R \in \mathbb{R}$) have this maximal number of Killing vectors.*

*Proof.*
We prove each of the three claims separately.

1. The proof is quite similar to the one for the uniqueness conditions on local isometries. This should be no surprise since it has already been shown how closely Killing vector fields and isometries are linked.
   So again we define $A$ as the set where $\xi$ and $\nabla\xi$ vanish. $A$ is obviously closed by continuity. We prove that $A$ is also open. Let $q \in A$ and $U_q$ be a normal neighbourhood of $q$. Let $\gamma$ be a geodesic through $q$. Then according to the last lemma $\xi_\gamma$ is a Jacobi field. Lemma 1.3 tells us that a Jacobi field is uniquely determined by its value and its derivative at $q$ and hence vanishes along the geodesic $\gamma$. Thus $\xi$ vanishes on $U_q$.

2. Let $\xi$ be a Killing vector field then the Killing equation (Remark 2.2) yields:

$$\xi_{a;b} + \xi_{b;a} = 0. \tag{2.6}$$

   Furthermore for $\xi$ as for every vector field we have

$$\xi_{a;b;k} - \xi_{a;k;b} = \sum_m R^m_{abk}\xi_m. \tag{2.7}$$

   By applying the symmetry properties of the Riemannian curvature tensor to equation (2.7) one gets

$$(\xi_{a;b} - \xi_{b;a})_{;k} + (\xi_{k;a} - \xi_{a;k})_{;b} + (\xi_{b;k} - \xi_{k;b})_{;a} = 0,$$

and because of (2.6) for Killing vector fields this equation reduces to

$$\xi_{a;b;k} + \xi_{k;a;b} + \xi_{b;k;a} = 0.$$

Using (2.6) and (2.7) this equations yields

$$\xi_{k;b;a} = \sum_m R^m_{abk} \xi_m.$$

3. Claim 1 tells us that a Killing vector field is uniquely determined by its value $\xi_p$ at a point $p \in M$ and the value of $\nabla\xi$ at $p$. From Theorem 2.1 we know that $\nabla\xi$ is a skew-adjoint operator. Therefore

$$\begin{aligned} & dim(i(M)) \\ \leq\ & (dim(T_pM) + dim(\{A \in gl(T_pM) | A \textit{ is skew-adjoint}\}) \\ =\ & n + \frac{n(n+1)}{2} = \frac{n(n+1)}{2}. \end{aligned}$$

So it remains to prove that only spaces of constant curvature can have that many Killing vectors:
As a tensor (with respect to a coordinate chart) $\xi^l_{;b}$ obeys the following equation:

$$\xi_{l;b;a;i} - \xi_{l;b;i;a} = \sum_m R^m_{lai} \xi_{m;b} + R^m_{bai} \xi_{m;l}.$$

After combining this with equation (2.5) and the Killing equation one obtains:

$$\sum_{k,m} (R^m_{abl;i} - R^m_{ibl,a}) \xi_m + (R^m_{abl} g^k_i - R^m_{ibl} g^k_a + R^m_{bai} g^k_l - R^m_{lai} g^k_b) \xi_{m;k} = 0.$$

In general this equation puts further restrictions on the shape of the Killing vector fields. If we want to avoid these restrictions, the tensor $R^a_{bcd}$ must be of such form that this equation is no more a restriction on the Killing vector field $\xi$. Thus due to the Killing equation $\xi_{m;k} = -\xi_{k;m}$ one has

$$R^m_{abs;i} = R^m_{ibs;a}, \tag{2.8}$$

$$\begin{aligned} & R^m_{abs} g^k_i - R^k_{abs} g^m_i - R^m_{ibs} g^k_a + R^k_{ibs} g^m_a \\ & \quad + R^m_{bai} g^k_s - R^k_{bai} g^m_s - R^m_{sai} g^k_b + R^k_{sai} g^m_b = 0. \end{aligned} \tag{2.9}$$

Contraction in (2.9) over $(i, k)$ yields

$$(n-1) R^m_{abs} = R_{as} g^m_b - R_{ab} g^m_s.$$

Another contraction over $(a, b)$ reveals:

$$nR^m_s = Rg^m_s.$$

Thus the tensor $R_{mabs}$ must be of the following form:

$$R_{mabs} = \frac{R}{n(n-1)}(g_{as}g_{mb} - g_{ab}g_{ms})$$

and due to equation (2.8) $R \in \mathbb{R}$ must be constant.

□

**Corollary 2.2.**
*Let $p \in M$ be a point on $M$ and $\xi \in \Gamma(TM)$ be a Killing vector field on $M$ such that $\xi_p = 0$. Then locally $\xi$ is tangential to the geodesic circles around $p$.*

*Proof.*
Since $\xi$ has constant angle with any geodesic passing through $p$ it must be orthogonal to all these geodesics: For any sufficiently small $s \in \mathbb{R}$ and geodesic $\gamma$ through $p$ one knows $g(\xi_{\gamma(s)}, \gamma'(s)) = g(\xi_p, \gamma'(0)) = 0$ since $\xi_p = 0$ by assumption. □

**Theorem 2.2.** "A Better Estimate for the Dimension of $i(M)$"
*For a Killing vector field $\xi$ on $M$ the following equations must hold:*

0) $L_\xi R_{abcd} = 0,$
1) $L_\xi(R_{abcd;i_1}) = 0,$
2) $L_\xi(R_{abcd;i_1;i_2}) = 0,$
...
$k$) $L_\xi(R_{abcd;i_1;i_2;\ldots;i_k}) = 0,$
... .

*At a point $p \in M$ each line can be regarded as a subsystem of linear equations with respect to the variables $\xi^i$ and $\xi^i_{;j}$. All lines together form a system of infinitely many equations. The number of linearly independent Killing vector fields on $M$ is* ***less or equal*** *to $r$ if the rank of the whole system (i.e. the maximal number of linearly independent equation) is*

$$\frac{n(n+1)}{2} - r.$$

*Proof.*
It is clear that the Riemannian curvature tensor and the L.-C. connection only depend on the metric of a manifold. Since along a Killing vector field the Lie derivative of the metric vanishes we also know that the Lie derivative of the curvature tensor must vanish. For the same reason the Lie derivative along a Killing vector field of any covariant derivative of $R_{abcd}$ must vanish as well. Thus the Killing vectors lie inside the system's kernel which we assume to be of dimension $r$. Linear algebra tells us that the dimension of the kernel is the difference between the dimension of the domain and the dimension of the image. According to Lemma 2.6 this is $\frac{n(n+1)}{2} - r$. The statement of the theorem can also be seen as a consequence of compatibility conditions for the corresponding system of differential equations, which can be found in [5]. □

**Remark 2.4.**
*The expression* $L_\xi R_{abcd}$ *can be written in terms of the "unknowns"* $\xi^a$ *and* $\xi_{a;i}$*,* $a, i \in \{1, ..., n\}$*. This makes clear how the equations in Theorem 2.2 can be interpreted as a system of linear equations with respect to the unknowns* $\xi^a$ *and* $\xi_{a;i}$ *at any point* $p \in M$*.*

$$\begin{aligned}
&L_\xi R_{ijkl} \\
&= L_\xi R(\delta_i, \delta_j, \delta_k, \delta_l) \\
&= \xi R_{ijkl} - R(L_\xi \delta_i, \delta_j, \delta_k, \delta_l) - R(\delta_i, L_\xi \delta_j, \delta_k, \delta_l) \\
&\quad - R(\delta_i, \delta_j, L_\xi \delta_k, \delta_l) - R(\delta_i, \delta_j, \delta_k, L_\xi \delta_l) \\
&= \xi R_{ijkl} - R(\nabla_\xi \delta_i, \delta_j, \delta_k, \delta_l) + R(\nabla_{\delta_i} \xi, \delta_j, \delta_k, \delta_l) - ... \\
&\quad - R(\delta_i, \delta_j, \delta_k, \nabla_\xi \delta_l) + R(\delta_i, \delta_j, \delta_k, \nabla_{\delta_l} \xi) \\
&= \sum_a \left( \xi^a R_{ijkl;a} + \xi^a_{;i} R_{ajkl} + \xi^a_{;j} R_{iakl} + \xi^a_{;k} R_{ijal} + \xi^a_{;l} R_{ijka} \right) \\
&= \sum_a \left( \xi^a R_{ijkl;a} + \xi_{a;i} R^a_{jkl} + \xi_{a;j} R_i{}^a{}_{kl} + \xi_{a;k} R_{ij}{}^a{}_l + \xi_{a;l} R_{ijk}{}^a \right).
\end{aligned}$$

We have already seen that Killing vector fields can be regarded as infinitesimal isometries since all stages of their flow are isometries. The next theorems will provide us with further information on the relation between isometries and Killing vectors.

**Definition 2.4.** "Action of a Lie Group on a Manifold"
*The **action** of a Lie Group* $G$ *on* $M$ *is a smooth map:* $G \times M \mapsto M$*,* $(g, p) \mapsto g(p)$ *such that*

1. *(ab)(p) = a(b(p)) for all* $a, b \in G$ *and* $p \in M$*,*

2. *e(p)=p for all* $p \in M$ *where e is the neutral element in the group G.*

**Lemma 2.7.** "$i(M)$ as a Lie Algebra"
*$i(M)$ denotes the set of all Killing vector fields on M. $(i(M), [.,.])$ is a finite-dimensional real Lie algebra (in fact a Lie subalgebra of the Lie algebra $\Gamma(TM)$) as a vector space over $\mathbb{R}$ (Here $[.,.]$ denotes the Lie bracket).*

*Proof.*
$V, W \in \Gamma(TM)$.

1. $L_V$ is $\mathbb{R}$-linear in $V$. Hence $i(M)$ is a $\mathbb{R}$-vector space.

2. $[L_V, L_W] = L_{[V,W]}$ hence $i(M)$ is closed with respect to the bracket operator.

3. The Lie bracket fulfils the properties of the bracket operator from Definition 1.19.

4. Lemma 2.6 shows that $i(M)$ is finite-dimensional.

□

**Theorem 2.3.** "$I(M)$ as a Lie Group"
*Since by linking two isometries one gets an isometry again, $I(M)$ is a group. There is a unique way to make $I(M)$ a manifold such that*

1. *$I(M)$ becomes a Lie group,*

2. *The action $G \times M \mapsto M$, $(f, p) \mapsto f(p)$ is smooth,*

3. *A homomorphism $\beta : \mathbb{R} \mapsto I(M)$ is smooth if the map $\mathbb{R} \times M \mapsto M$, $(t, p) \mapsto \beta(t)(p)$ is smooth.*

*Proof.*
The proof and the neccessary theoretical backround of this theorem can be found in [14, pp. 90-105]. □

**Remark 2.5.**
*A translation on $M$ in direction of a Killing vector field can be interpreted as a motion (e.g. a rotation) of the space. Therefore one has the following notation: $I(M)$ is often referred to as the* ***group of motions*** *of $M$. $i(M)$ can be regarded as the* ***transformation group*** *of $M$ and is often denoted by $G_n$.*

**Theorem 2.4.** "Killing Vector Fields and Isometries"
*Let $T_{id}(I(M))$ be the Lie algebra of $I(M)$ and $i(M)^c$ be the set of all* ***complete*** *Killing vector fields on $M$.*

1. *$i(M)^c$ is a Lie subalgebra of $i(M)$.*
2. *$i(M)^c$ is isomorphic to $T_{id}(I(M))$.*

*Here id denotes the identity map.*

*Proof.*
To proof claim (2) one has to find a linear map between $T_{id}(I(M))$ and $i(M)^c$ and show that it is **injective** and **surjective**. Define

$$\beta_p: \ I(M) \mapsto M, \ f \mapsto f(p).$$

Since Theorem 2.3 ensures the smoothness of actions, the map $\beta_p$ must be smooth as well.

Let $X_{id} \in T_{id}(I(M))$ be a tangent vector in the Lie algebra of $I(M)$ and $\alpha : \mathbb{R} \mapsto I(M)$ the corresponding one-parameter group as explained in Lemma 1.8. Then

$$D\beta_p : T_{id}(I(M)) \mapsto T_pM, \ X_{id} \mapsto D\beta_p(X_{id}) =: \tilde{X}_p$$

is a linear map that gives rise to the vector field $\tilde{X} \in \Gamma(TM)$.
This allows to define the linear map

$$D\beta : T_{id}(I(M)) \mapsto \Gamma(TM), \ X_{id} \mapsto D\beta_{(.)}(X_{id}) =: \tilde{X}_{(.)}$$

and we will show that $D\beta$ is an isomorphism between $T_{id}(I(M))$ and $i(M)^c$. Furthermore one has the curve $\gamma_p := \beta_p \circ \alpha$ on $M$ which passes through $p \in M$ such that

$$\tilde{X}_p = \gamma_p'(0) = D\beta_p(\alpha'(0)).$$

First we prove that $D\beta$ is **injective**:

For $\tilde{X} = 0$ one has to show that $X_{id}$ must be zero. This can be done by showing that $\alpha(t)$ is constant. $\tilde{X} = 0$ means that $\tilde{X}_p = 0$ and $\tilde{X}_{\beta_p(\alpha(s))} = 0$ for every $p \in M$, $s \in \mathbb{R}$.
This yields:

$$\begin{aligned}
0 = \tilde{X}_{\beta_p(\alpha(s))} &= D\beta_{\beta_p(\alpha(s))}(X_{id}) = D\beta_{\beta_p(\alpha(s))}(\alpha'(0)) \\
&= \frac{d}{dt}|_0 \alpha(t) \circ \alpha(s)(p) = \frac{d}{dt}|_0 \alpha(s+t)(p) = \frac{d}{dt}|_0 \beta_p \circ \alpha(s+t) \\
&= \frac{d}{dt}|_s \beta_p \circ \alpha(t) = D\beta_p(\alpha'(s)) = \gamma'(s).
\end{aligned}$$

Thus for every $p \in M$ $\gamma_p$ is a constant curve namely $\gamma_p(t) = p$. Using the definition $\gamma(t) := \beta_p \circ \alpha(t) := \alpha(t)(p)$ one gets

$$\alpha(t)(p) = p \quad for\ all\ t \in \mathbb{R}.$$

This means that $\alpha(t)$ is the identity map for every $t \in \mathbb{R}$ and thus a constant one-parameter group. The unique corresponding element of the Lie algebra is the vector $X_{id} = 0$. This proves that $D\beta$ is injective.

Next we show that $D\beta$ is **surjective**:
Let $Y \in i(M)^c$ be a complete Killing vector field on $M$. We have to find a matching $X_{id} \in T_{id}(I(M))$ such that $\tilde{X} = D\beta(X_{id}) = Y$.
Let $\phi_t$ be the local flow of $Y$. Since $Y$ is a Killing vector field, $\phi_t$ is an isometry for every $t \in \mathbb{R}$. This means that $\phi_t$ is a one-parameter group in $I(M)$. Now choose $X_{id} := \frac{d}{dt}|_0 \phi_t$. Then one has

$$D\beta_p(X_{id}) = \frac{d}{dt}|_0 \phi_t(p) = Y_p.$$

This proves that $\beta$ is surjective. □

## 2.2 Examples

**Example 2.2.** "Killing Vectors on $\mathbb{R}^n_m$"
*$\mathbb{R}^n_m$ is a space of constant curvature. From Theorem 2.6 we know that it must have $\frac{n(n+1)}{2}$ Killing vectors. One can check that translations are generated by Killing vectors and that skew-adjoint linear transformation are generated by Killing vectors. Thus we already know all Killing vectors: There are always $n$ linearly independent translations and $\frac{n(n-1)}{2}$ skew-adjoint operators. We have already used this fact to find the maximal number of Killing vectors in Theorem 2.6. This means that $\mathbb{R}^n_m$ is maximally symmetric. These Killing vectors can be found by solving the system of differential equations imposed by the Killing equation from Remark 2.2 they are described by: In this case the Killing equation can be reduced to*

$$\xi_{a,b} + \xi_{b,a} = 0.$$

*Differentiating these equations yields*

$$\xi_{a,bc} + \xi_{b,ac} = 0,\ \xi_{b,ca} + \xi_{c,ba} = 0 \text{ and } \xi_{c,ab} + \xi_{a,cb} = 0.$$

*Altogether one gets*

$$\xi_{a,bc} = 0.$$

*Therefore the solution must be of the form*

$$\xi_a = \alpha_a + \sum_b \beta_{ab} x^b \text{ with } \alpha_a, \beta_{ab} \in \mathbb{R}$$

*and the symmetry $\beta_{ab} = -\beta_{ba}$ because of the Killing equation.*
*In general relativity theory one usually deals with $\mathbb{R}^4_1$ with 10 linearly independent Killing vectors:*
*4 translations (represented by the constants $\alpha_a$) and 6 rotations (represented by the constants $\beta_{ab}$).*

One can already see that there are several approaches to find Killing vector fields. $\mathbb{R}^n_m$ of course is an easy case since it has the maximal number of Killing vectors. This example is still important since it is the foundation upon which the manifolds in general relativity theory are build.

**Example 2.3.** "Schwarzschild Halfplane"
*The* ***Schwarzschild Halfplane*** *$P_I$ is the manifold $\mathbb{R} \times \mathbb{R}^+$ with coordinates $(t, r)$ and the metric*

$$g = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})dr^2.$$

*This is an essential tool for the basic idea on how to model the spacetime around a star or a black hole. A map*

$$\phi : P_I \to P_I,\ (t,r) \to (\phi^t(t,r), \phi^r(t,r))$$

*is an isometry if and only if*

$$g = D\phi^* g. \tag{2.10}$$

*So all we have to do is compute $D\phi^* g$ and compare its components to those of $g$.*

$$\begin{aligned} D\phi^* g = &\left( (\frac{2GM}{\phi^r(t,r)} - 1) \left(\frac{\partial \phi^t}{\partial t}\right)^2 + (1 - \frac{2GM}{\phi^r(t,r)}) \left(\frac{\partial \phi^t}{\partial r}\right)^2 \right) dt^2 \\ &+ 2\left( (\frac{2GM}{\phi^r(t,r)} - 1) \frac{\partial \phi^t}{\partial t}\frac{\partial \phi^r}{\partial t} + (1 - \frac{2GM}{\phi^r(t,r)}) \frac{\partial \phi^r}{\partial r}\frac{\partial \phi^t}{\partial r} \right) dtdr \\ &+ (\frac{2GM}{\phi^r(t,r)} - 1) \left( \left(\frac{\partial \phi^r}{\partial t}\right)^2 + \left(\frac{\partial \phi^r}{\partial r}\right)^2 \right) dr^2. \end{aligned}$$

*Equation 2.10 then yields:*

$$\begin{aligned} &\frac{\partial \phi^t}{\partial r} = \frac{\partial \phi^r}{\partial t} = 0, \\ &\frac{\partial \phi^t}{\partial t} = \frac{\partial \phi^r}{\partial r} = 1, \\ &\phi^r(t,r) = r. \end{aligned}$$

*Therefore isometries are translations in direction of the t-coordinate. The stages of the flow of a Killing vector field are isometries. Therefore $P_I$ has only $\frac{\partial}{\partial t}$ as a Killing vector.*

**Example 2.4.** "Killing Vectors of the 2-Sphere"
*The metric of the 2-sphere with coordinates $(\theta, \phi)$ is given by*

$$d\theta^2 + \sin^2(\theta) d\phi^2.$$

*The Killing equations from Remark 2.2 for $\theta = x_1$ and $\phi = x_2$ then read as*

$$\xi^1_{,1} = 0, \tag{2.11}$$

$$\xi^1_{,2} + \sin^2(\theta)\xi^2_{,1} = 0, \tag{2.12}$$

$$\xi^1 \cos(\theta) + \sin(\theta)\xi^2_{,2} = 0. \tag{2.13}$$

*Now we combine these equations to calculate the Killing vectors:*

$$
\begin{aligned}
(2.11) &\Rightarrow \xi^1(\theta,\phi) = \xi^1(\phi). \\
(2.13) &\Rightarrow \xi^2_{,\phi} = -\cot(\theta)\xi^1(\phi) \\
&\Rightarrow \xi^2(\theta,\phi) = -\cot(\theta)F(\phi) + h(\theta) \textit{ where } F'(\phi) = \xi^1(\phi) \\
&\Rightarrow \xi^2_{,\theta} = \frac{F(\phi)}{\sin^2(\theta)} + h'(\theta).
\end{aligned}
$$

*Substituting this into (2.12) yields:*

$$
\begin{aligned}
&\xi^1_{\phi}(\phi) + F(\phi) + h'(\theta) = 0 \Leftrightarrow F''(\phi) + F(\phi) = -h'(\theta) \\
\Rightarrow\ & h'(\theta) = 0 \textit{ and } F(\phi) = A\sin(\phi + a) + B\cos(\phi + b) \\
\Rightarrow\ & \xi^1(\phi) = F'(\phi) = A\cos(\phi + a) - B\sin(\phi + b) \\
\Rightarrow\ & \xi^2(\phi) = -\cot(\theta)\left(A\sin(\phi + a) + B\cos(\phi + b)\right) + c
\end{aligned}
$$

*for* $A, B, a, b, c \in \mathbb{R}$.
*It is due to the trigonometric addition theorems, that only the three parameters* $A$*,* $B$ *and* $c$ *contribute to a basis:*
*After setting* $A = 1$*,* $B = 0$*,* $c = 0$ *one gets*

$$
\begin{aligned}
& \begin{pmatrix} \cos(\phi + a) \\ -\cot(\theta)\sin(\phi + a) \end{pmatrix} \\
= \ & \cos(a)\begin{pmatrix} \cos(\phi) \\ -\sin(\phi)\cot(\theta) \end{pmatrix} - \sin(a)\begin{pmatrix} \sin(\phi) \\ \cos(\phi)\cot(\theta) \end{pmatrix}.
\end{aligned}
$$

*Thus there are three Killing vectors each of which we obtain by setting one parameter to* $1$ *and the others to* $0$*:*

$$
\xi_1 = \begin{pmatrix} \sin(\phi) \\ \cos(\phi)\cot(\theta) \end{pmatrix},\ \xi_2 = \begin{pmatrix} \cos(\phi) \\ -\sin(\phi)\cot(\theta) \end{pmatrix},\ \xi_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

*Note that the lower index in this case does not denote the covariant component but simply distinguishes the elements of the basis given above.*

On this picture one sees a visualization of the Killing vector field $\xi_3$ and its integral curves on the sphere. Every vector field on a sphere must vanish at one point. Corollary 2.2 tells us that a Killing vector field must be tangent to geodesic circles around that point. Since one knows what the shape of geodesic circles on a sphere is, one could have guessed without any calculations what the Killing vector fields of a sphere must look like.

## 2.3 Classification of $G_2$

As already mentioned in the historical note the classification of Lie algebras has been of great interest to many mathematicians. As we will see in Chapter 3 the structure of the transformation group of a space is of great importance in general relativity theory. In the next sections we will shortly go through the approaches for the classification of $G_2$, $G_3$ and $G_4$.

**Definition 2.5.** "Structure Constants Vector"
*The* ***structure constants vector*** $c = (c_1, ...., c_n)^T$ *is defined as*

$$c_b := \sum_a C^a_{ab}.$$

The structure constants vector is used to classify Lie algebras. In the case of $G_2$ it even completely classifies a given algebra:
Let $\{e_1, e_2\}$ be a basis of $G_2$.
Due to Definition 1.20 we know that $C^k_{ii} = 0$ and $C^k_{12} = -C^k_{21}$, $i, j, k \in \{1, 2\}$.
Case 1: $c = 0$.
This is the abelian case where all structure constants vanish with

$$[e_1, e_2] = 0.$$

Case 2: c$\neq$ 0.
In this case we have $c = (C^2_{21}, C^1_{12})$ due to $[e_1, e_2] = C^1_{12}e_1 + C^2_{12}e_2$.
The basis transformation

$$e'_1 = C^1_{12}e_1 + C^2_{12}e_2,$$
$$e'_2 = \frac{1}{C^1_{12}}e_2$$

provides the canonical form

$$[e'_1, e'_2] = e'_1.$$

This means that every 2-dimensional Lie algebra is isomorphic to the first or the second case. This approach can be extended to the classification of higher dimensional Lie algebras such as the $G_4$ in this chapter.

## 2.4 Bianchi Classification of $G_3$

Bianchi's approach to classify $G_3$ was similar to the one presented in this chapter for the classification of $G_2$ and $G_4$. The approach in this section can be considered as a modernized version of Bianchi's work and thus is a bit different. In [1, pp. 29-32] one can find even another approach that is similar to the one in this section.
Let $\{e_1, e_2, e_3\}$ be a basis of $G_3$. Since $C_{ij}^m = -C_{ji}^m$ we can define the $3 \times 3$ matrix $N$, containing all information about the structure constants as

$$N = \begin{pmatrix} C_{23}^1 & C_{13}^1 & C_{12}^1 \\ C_{23}^2 & C_{13}^2 & C_{12}^2 \\ C_{23}^3 & C_{13}^3 & C_{12}^3 \end{pmatrix}.$$

More rigorously defined:

$$N_{de} := \sum_{AB=1}^{3} \frac{1}{2} \Delta^{ABe} C_{AB}^d, \; d, e \in \{1, 2, 3\}.$$

$\Delta^{ijk}$ is the Levi-Civita symbol, i.e. $\Delta^{12\dots n} = 1$ and switching indices switches sign.
This matrix can be decomposed into its symmetric component $\tilde{n}$ and skew-symmetric component $\tilde{\alpha}$ which is represented by a vector $\tilde{a}$:

$$N_{ij} = \tilde{n}_{ij} + \Delta^{ijk} a_k.$$

$$\tilde{n} = \begin{pmatrix} n_1 & n_4 & n_6 \\ n_4 & n_2 & n_5 \\ n_6 & n_5 & n_3 \end{pmatrix} \alpha = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} with \; \tilde{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

For the constants $C_{ij}^k$ this yields:

$$C_{ij}^k = \sum_m \Delta^{mij} \tilde{n}_{km} + \delta_j^k a_i - \delta_i^k a_j.$$

By substituting these formulas into the Jacobi identity from Definition 1.20 one obtains:

$$\sum_i \tilde{n}_{ij} a_i = 0.$$

One can always find a basis of $G_3$ such that $\tilde{a} = (a, 0, 0)$, $a \in \mathbb{R}$, and with a principal components analysis of $\tilde{n}$ one transforms $\tilde{n}_{ii} =: n_i$ into $+1, -1$ or $0$.

Therefore each group in $G_3$ is classified by its constants $(n_1, n_2, n_3, a)$ and can brought into the following form:

$$[e_1, e_2] = n_3 e_3 + a e_2,$$

$$[e_2, e_3] = n_1 e_1,$$

$$[e_3, e_1] = n_2 e_2 - a e_3$$

where $a \cdot n_1 = 0$ and $n_i \in \{-1, +1, 0\}$.

The Bianchi classification enumerates these nine cases for $(n_1, n_2, n_3, a)$ in the following way:

| | **Bianchi-Type** | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | **I** | **II** | **VII$_0$** | **VI$_0$** | **IX** | **VIII** | **V** | **IV** | **III** | **VII** | **VI** |
| $a$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | $a$ | $a$ |
| $n_1$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| $n_2$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| $n_3$ | 0 | 1 | 1 | $-1$ | 1 | $-1$ | 0 | 1 | $-1$ | 1 | $-1$ |

Of course the cases VII and VI represent a whole one-parameter family of non-isomorphic groups with parameter $a \in \mathbb{R}$. The cases with $a = 0$ have been denoted separately by $VII_0$ and $VI_0$.

**Example 2.5.** "Classification of some Lie Algebras."

1. Flat Space $\mathbb{R}^3$
   *In Example 2.2 we have seen that there are six Killing vectors in* $\mathbb{R}^3$*:*

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}y\\-x\\0\end{pmatrix}, \begin{pmatrix}z\\0\\-x\end{pmatrix}, \begin{pmatrix}0\\z\\-y\end{pmatrix}.$$

   *The first three vectors are the basis of an abelian Lie subalgebra since all brackets vanish. This is* ***Bianchi type I****. The last three vectors, which generate the rotations on* $\mathbb{R}^3$*, are the basis of a 3-dimensional subalgebra of Bianchi Type* ***IX****.*

2. Group of Rotations in $\mathbb{R}^3$
   *From Example 1.3 we know that the group of rotations in* $\mathbb{R}^3$*,* $SO(3)$*, has the Lie algebra* $so(3)$ *with the basis*

$$a_1 = \begin{pmatrix}y\\-x\\0\end{pmatrix}, \quad a_2 = \begin{pmatrix}z\\0\\-x\end{pmatrix}, \quad a_3 = \begin{pmatrix}0\\z\\-y\end{pmatrix}.$$

*The bracket operator in this case is the commutator:*

$$[A, B] = AB - BA \text{ for } A, B \in so(3).$$

*One has*

$$[a_1, a_2] = a_3, \quad [a_2, a_3] = a_1, \quad [a_1, a_3] = a_2.$$

*This is* ***Bianchi type IX****.*

3. The 2-Sphere
*From Example 2.4 we know the three Killing vectors of the 2-sphere:*

$$e_1 = \begin{pmatrix} \sin(\phi) \\ \cos(\phi)\cot(\theta) \end{pmatrix}, e_2 = \begin{pmatrix} \cos(\phi) \\ -\sin(\phi)\cot(\theta) \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

*The commutators are:*

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_3] = e_2.$$

*This is* ***Bianchi type IX****. Thus it is the same Bianchi type as the Lie algebra of the rotation group in flat* $\mathbb{R}^3$*. This should be no surprise since the 2-sphere is a transitivity area of the group of rotations, i.e. a point on the 2-sphere can by mapped into any other point on the 2-sphere by a rotation.*

## 2.5 Classification of $G_4$

There are several approaches for the classification of $G_4$. In fact they are not that different and MacCallum compares some of these enumerations made by Lie, Kruchkovich, Patera and others in his paper "On the Classification of the Real 4-dimensional Lie Algebras" in [9]. In this paper MacCallum also goes into detail in comparing the other approaches and gives much of the historical background of these enumerations. This section gives an introduction to the approach that is used to classify 4-dimensional Lie algebras, which MacCallum denotes by $L_4$. The details are explained very comprehensively in MacCallum's paper.

First we have to remember the **structure constants vector** $c$ given by

$$c_b := \sum_a C^a_{ab}, c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

The idea is to distinguish two classes of algebras: The unimodular ones ($c = 0$) and the non-unimodular ones ($c \neq 0$). The Jacobi identity from Definition 1.20 implies

$$\sum_i c_i C^i_{ab} = 0. \tag{2.14}$$

First we consider the **non-unimodular** case:
This class is denoted by $N$. After an appropriate basis transformation $c$ is of the form $(0,0,0,\gamma)^T$(MacCallum uses "$c$" instead of $\gamma$). Then (2.14) yields that

$$C^a_{bc} \; and \; C^a_{b4} =: \theta^a_b \; for \; a,b,c \in \{1,2,3\}$$

remain to be classified. The first group of structure constants can be written in a matrix form:

$$n^{ab} := \sum_{e,f=1}^{4} \frac{1}{2\gamma} C^a_{de} \Delta^{debf} c_f, \;\; a,b \in \{1,2,3,4\}. \tag{2.15}$$

This yields that $n^{ab}$ is symmetric and $n^{a4} = 0$. By applying a principle component analysis one can find a subspace $V = span(e_1, e_2, e_3)$ such that the matrix $n^{ab}$ becomes a diagonal matrix with diagonal elements $(n_1, n_2, n_3, 0)$, $n_i \in \{1, -1, 0\}$. The Jacobi identity from Definition 1.20 then yields

$$n_1(2\theta^1_1 - \gamma) = n_2(2\theta^2_2 - \gamma) = n_3(2\theta^3_3 - \gamma) = 0, \tag{2.16}$$

$$n_2\theta_2^3 + n_3\theta_3^2 = n_3\theta_3^1 + n_1\theta_1^3 = n_1\theta_1^2 + n_2\theta_2^1 = 0. \tag{2.17}$$

The matrix $n^{ab}$ cannot have rank 3 since this leads to a contradiction with

$$\gamma = \theta_i^i \; for \; i \; \in \{1, 2, 3\}.$$

In the case $rank(n^{ab}) = 2$ there are two subcases according to the signature of the matrix.
In the case $rank(n^{ab}) = 1$ it is possible to find a basis such that

$$\theta_1^1 = \gamma/2 \neq 0 \; for \; \gamma = 4 \; and \; \theta_2^1 = \theta_1^2 = \theta_3^1 = \theta_1^3 = 0.$$

Thus only the matrix $\theta_j^i \; for \; i, j \in \{1, 2\}$ remains to be classified which is done by its Segre Type. The Segre type is a list $[s1, s2, ..., s_k]$ of the dimensions of the Jordan blocks. If distinct Jordan blocks have equal eigenvalues the corresponding digits are enclosed inside round brackets. The last case would be $rank(n) = 0$ where the matrix $\theta_l^k \; for \; k, l \in \{1, 2, 3\}$ is to be classified by its Segre Type as well. This approach explains the notation used for the enumeration:
First we put the Letter $N$ (for non-unimodular), followed by the rank of the matrix $n^{ab}$ (i.e. $0, 1$ or $2$).
In class $N2$ the last digit is the modulus of the signature of $n^{ab}$, in $N1$ and $N0$ the last digit stands for the Segre Type of the remaining matrix $\theta_j^i$.

Now we deal with the **unimodular** case ($c = 0$).
To distinguish the unimodular classes we will use a theorem by Farnsworth and Kerr:

**Theorem 2.5.**
*For an unimodular algebra $G_4$ either there is a vector $p \in \mathbb{R}^4$ such that*

$$C_{bd}^a = \theta_b^a p_d - \theta_d^a p_b \; , \; \sum_a \theta_b^a p_a = 0, \tag{2.18}$$

*or if there is no such vector $p$, then there is a nonzero vector $l \in \mathbb{R}^4$ such that:*

$$\sum_d C_{bd}^a l^d = 0. \tag{2.19}$$

*Proof.*
The proof can be found in MacCallum's paper [9, pp. 303-304] or in [6]. Here only the basic idea behind the proof is explained shortly. One defines $C^a$ to

be the 2-form with components $C^a_{bc}$ with respect to a basis in $G_4$. Then the Jacobi identities yield

$$C^i \wedge C^j = 0.$$

The case $i = j$ shows that $C^a$ is a **simple** 2-form and therefore can be interpreted as a plane. (A 2-form $\eta$ is called "simple" if it can be written as $\eta = dx^i \wedge dx^j$ in a coordinate system $x_i$. Thus one can regard $dx^i$ and $dx^j$ as a basis of a plane.)
The case $i \neq j$ shows that these planes must intersect in lines. The statement of the theorem is then proved mainly by considering the different ways in which the four planes $C^a,\ a \in \{1, ..., 4\}$ can intersect each other. □

**Corollary 2.3.**
*Every real 4-dimensional Lie algebra contains an invariant 3-dimensional Lie algebra.*

*Proof.*
A proof is given in MacCallum's paper [9, pp. 304]. □

This leads to the following terminology: The unimodular Lie algebras are divided into two main groups:

- $U1$: The plains of the 2-forms in the proof meet in one line.
- $U3$: The plains of the 2-forms in the proof meet at least in three lines.

$U1$ can be regarded as a limit of the class $N0$ and in the same manner the matrix $\theta^a_b\ for\ a, b \in \{1, 2, 3\}$ remains to be classified which is done by its Segre Type.
Subclassification of $U3$ is done by choosing a basis such that $l^a = (0, 0, 0, 1)$. Then one can have the case $DG_4 = V$ which means that $V$ is semi-simple (and simple). The class is called $U3S$ ("S" stands for "simple").
The other possibility is that one has $dim(DG_4 \cap V) = 2$ and $G_4$ is integrable, so we call this class $U3I$ ("I" stands for "integrable"). Both classes $U3S$ and $U3I$ have subclasses determined by the signature of the matrix

$$n^{ab} := \frac{1}{2} \sum_{cd} \epsilon^{acd} C^b_{cd}$$

with a non-vanishing 3-form $\epsilon$.
The modulus of this signature is denoted at the end of the name of each class.

**Remark 2.6.** "Enumeration of the Distinct $G_4$"
*In [9, pp. 311-312] MacCallum gives a complete list of all classes and compares the different notations used by other mathematicians. Here a list only using MacCallum's notation is given. The main class* $U3I$ *can be seen as the limit of the class* $N1[1,1]$ *for* $c = 0$*. Class* $U1$ *can be seen as the limits of class* $N0$*.*

| ***Non-Unimodular Cases*** | ***Unimodular Cases*** |
|---|---|
| $N22$ | $U1[1,1,1]$ |
| $N20$ | $U1[1,1,1]_{\nu=0}$ |
| $N1[1,1]$ | $U1[(1,1,1)]$ |
| $N1[1,1]_{2,0}$ | $U1[Z,\bar{Z},1]$ |
| $N1[Z,\bar{Z}]$ | $U1[Z,\bar{Z},1]_{\lambda=0}$ |
| $N1[2]$ | $U1[2,1]$ |
| $N0[1,1,1]_{1,\mu,\nu}$ | $U1[(2,1)]$ |
| $N0[1,1,1]_{1,\mu,0}$ | $U1[3]$ |
| $N0[1,1,1]_{1,1,0}$ | $U3I0$ |
| $N0[1,1,1]_{1,0,0}$ | $U3I2$ |
| $N0[Z,\bar{Z},1]$ | $U3S1$ |
| $N0[Z,\bar{Z},1]_{\mu=0}$ | $U3S3$ |
| $N0[2,1]$ | |
| $N0[2,1]_{\lambda=1,\mu=0}$ | |
| $N0[2,1]_{\lambda=0,\mu=1}$ | |
| $N0[(2,1)]$ | |
| $N0[3]$ | |

# Chapter 3

# Killing Vectors in General Relativity Theory

The aim of this chapter is to explain which role Killing vectors play in general relativity. Cosmological models are meant to explain cosmological phenomena in a mathematically rigorous way. For instance one has a cosmological model for a star and its gravitational field. In order to understand how several galaxies influence each other one uses a different model.
We will see that Killing vectors are used to model certain observable, spatial properties of our universe in cosmological models. Furthermore Killing vectors are also helpful to construct convenient coordinates due to the symmetries they generate. Of course there are many more applications of Killing vectors but the ones presented in this chapter should provide a good intuition for the fields of application where Killing vectors are a helpful tool.

## 3.1 Birkhoff's Theorem

The Schwarzschild metric implies the assumption of a spherically symmetric spacetime in a vacuum. **Spherically symmetric** means that the Killing vectors of the spacetime are of the same Bianchi type as $so(3)$, that is Bianchi Type IX as we have seen in Example 2.4. For our further argumentation the **Froebenius Theorem** is crucial:

**Definition 3.1.** "Integrability"
*Let $M$ be a semi-Riemannian Manifold. A* ***subbundle*** *$E \subset TM$ of the tangent bundle $TM$ is* ***integrable****, if for any two (local) vector fields $V$ and $W$ lying in $E$, the Lie bracket $[V, W]$ lies in $E$ as well:*

$$V, W \in E \Rightarrow [V, W] \in E.$$

**Definition 3.2.** "Involutive Subbundle"
*A subbundle $E \subset TM$ of the tangent bundle $TM$ is **involutive**, if for every $p \in M$ there exists a submanifold $N \subset M$ such that*

$$E_p = T_p N.$$

**Theorem 3.1.** "Froebenius Theorem"
*For a subbundle $E \subset TM$ on a manifold $M$ the following equivalence holds:*

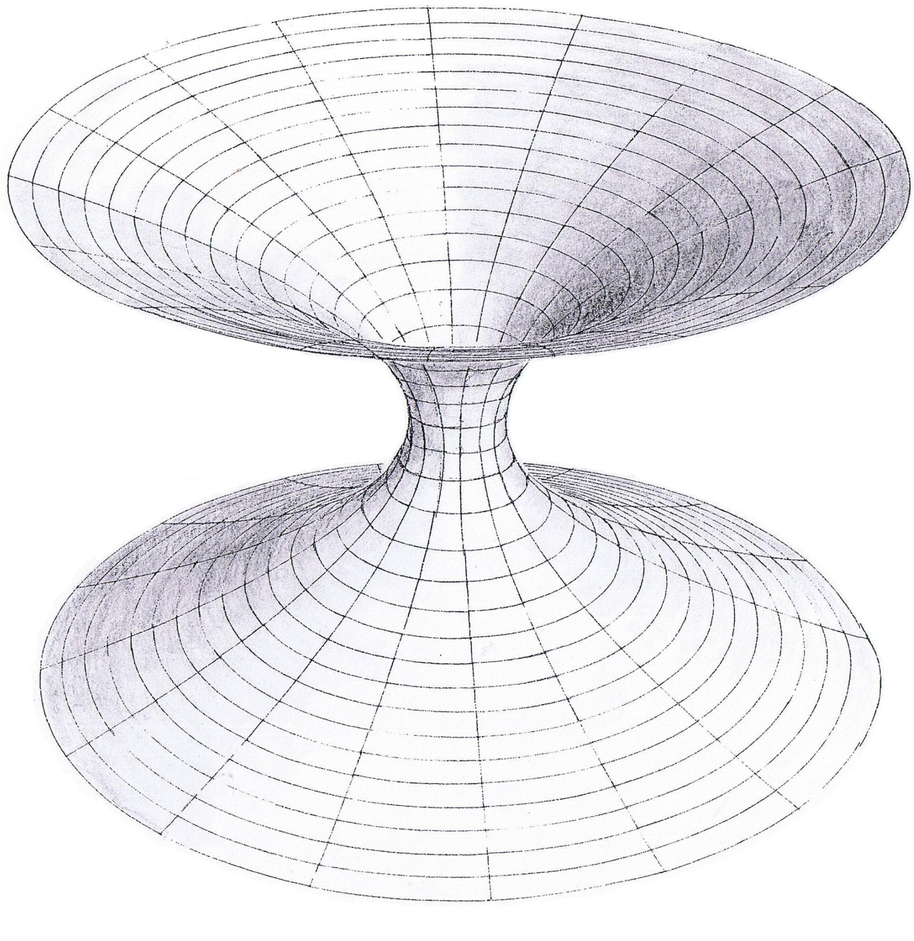
$$E \textit{ is integrable } \Leftrightarrow E \textit{ is involutive.}$$

*Proof.*
The proof as well as the necessary theoretical background can be found in [20, p. 113]. □

This theorem basically says that if one has a set of commuting vector fields then there exists a set of coordinate functions such that the vector fields are the partial derivatives with respect to these functions. If one has some vector fields which do not commute, but form a group (with the Lie bracket), then the integral curves of these vector fields "fit together" and describe submanifolds of the manifold on which they are all defined. Such a family of submanifolds is called a **foliation** of the manifold. This is a quite pictorial claim.

**Example 3.1.** "$\mathbb{R}^3$ foliated with 2-spheres"
$\mathbb{R}^3$ *is spherically symmetric as we have already seen in Example 2.5. Therefore it can be foliated by spheres. The drawing below shows some of these spheres with a Killing vector field tangent to these submanifolds. The foliation is not complete though, since the origin does not move on a sphere under rotations. At least almost the whole* $\mathbb{R}^3$ *is foliated, which suffices for our purpose.*

**Example 3.2.** "The Wormhole"
*In the last example the spheres were all centered around the origin. This is not necessarily always the case. A 2-dimensional wormhole is a good example: Topologically being of the structure* $\mathbb{R} \times S^1$ *it allows a foliation with 1-spheres (2-spheres in the 3-dimensional case) which are not centered around a certain point. The picture below shows a 2-dimensional wormhole in* $\mathbb{R}^3$ *with some of the circles foliating it.*



In general there is the following theorem:

**Theorem 3.2.**
*Let $M$ be a semi-Riemannian manifold that can be foliated by $m$-dimensional submanifolds, that are maximally symmetric. Then there is a coordinate system* $(\underbrace{v^1, ..., v^{n-m}}_{=:v}, \underbrace{u^1, ..., u^m}_{:=u})$ *such that the metric is of the form*

$$g = \sum_{i,j=1}^{n-m} \hat{g}_{ij}(v) dv^i dv^j + \hat{r}(v) \sum_{k,l=1}^{m} \tilde{g}_{kl}(u) du^k du^l.$$

*Proof.*
The theorem and its proof can be found in [19, pp. 395-401]. Here only a sketch of the proof for $\mathbb{R}^4$ with the 2-spheres as maximally symmetric submanifolds is presented, because this case provides the setting for Birkhoff's theorem.
Assuming that the space-like 2-spheres are maximally symmetric submanifolds suggests the use of generalized polar coordinates:

$$(x^1, x^2, x^3, x^4) = (t, r, \theta, \phi).$$

One obtains the three Killing vectors $\xi_1$, $\xi_2$, $\xi_3$ on the 2-sphere:

$$\xi_1 = \begin{pmatrix} 0 \\ 0 \\ \sin(\phi) \\ \cot(\theta) cos(\phi) \end{pmatrix}, \; \xi_2 = \begin{pmatrix} 0 \\ 0 \\ -\cos(\phi) \\ \cot(\theta)\sin(\phi) \end{pmatrix}, \; \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

So far no specifications for the metric $g$ have been made. The key idea is to use the Killing equation from Remark 2.2,

$$L_\xi(g_{ij}) = \sum_{k=1}^{n} g_{ij,k}\xi^k + g_{kj}\xi^j_{,i} + g_{ik}\xi^k_{,j} = 0,$$

to derive some properties of the metric.
Applying the Killing equation to $\xi_3$ yields

$$g_{ij,\phi} = 0.$$

Similarly by applying the Killing equation to $\xi_1$ and $\xi_2$ one gets

$$g_{11,\theta} = g_{00,\theta} = g_{10,\theta} = 0.$$

Furthermore the Killing equation for $\xi_1$ implies

$$g_{22,\theta}\sin(\phi) = 2g_{23}\frac{\cos(\phi)}{\sin^2(\theta)},$$

$$(g_{33,\theta} - 2g_{33}\cot(\theta))\sin(\phi) = -2g_{23}\cos(\phi),$$

$$g_{12,\theta}\sin(\phi) = g_{13}\frac{\cos(\phi)}{\sin^2(\theta)},$$

$$(g_{13,\theta} - g_{13}\cot(\theta))\sin(\phi) = -g_{12}\cos(\phi),$$

$$(g_{23,\theta} - g_{23}\cot(\theta))\sin(\phi) = \left(-g_{22} + g_{33}\frac{1}{\sin^2(\theta)}\right)\cos(\phi),$$

$$g_{20,\theta}\sin(\phi) = g_{30}\frac{\cos(\phi)}{\sin^2(\theta)},$$

$$(g_{30,\theta} - g_{30}\cot(\theta))\sin(\phi) = -g_{20}\cos(\phi).$$

The Killing equation for $\xi_2$ yields seven similar equations. Altogether one gets:

$$g_{00} = g_{00}(r,t),\ g_{11} = g_{11}(r,t),\ g_{22} = g_{22}(r,t),$$

$$g_{33} = g_{22}(r,t)\sin^2(\theta)\ \textit{and}\ g_{01} = g_{01}(r,t).$$

All the other components of $g$ vanish. The metric must then be of the form

$$\hat{g}_{11}(r,t)dt^2 + \hat{g}_{22}(r,t)dr^2 + \hat{g}_{12}(r,t)drdt + \hat{r}(r,t)(d\theta^2 + \sin^2(\theta)d\phi^2).$$

□

Basically this theorem says that up to a scaling factor $\hat{r}(v)$ all the submanifolds look the same. They have the coordinates $(u^1, ..., u^m)$ and they are lined up along an orthogonal manifold with coordinates $(v^1, ..., v^{n-m})$ (orthogonal, since there are no cross terms of the form $du^i dv^k$). One could say that the coordinates $v^i$ tell us on which submanifold we are, while the coordinates $u^i$ tell us where on a submanifold we are. This theorem also applies to the last two examples. As we have seen the spheres in $\mathbb{R}^3$ are maximally symmetric and it is the radius that plays the role of $\hat{r}(v)$ scaling the metric up or down. Similarly one can find such coordinates for a wormhole.
This concept also explains the idea behind the two terms stationary and static:
A spacetime is called **stationary** if it has a time-like Killing vector field. If this time-like Killing vector field is even orthogonal to a family of hypersurfaces then the spacetime is called **static**. The terminology is due to the fact that a stationary spacetime can be seen as one where things can only move in a symmetric way: The existence of a time-like Killing vector field implies that the field, along which the metric does not change, can have a time and a space component. Therefore the transitivity areas have a space component. This means, while following the flow of the Killing vector field,

things will actually move in space, but only in a symmetric way since it is an infinitesimal isometry. In the static case the time-like Killing vector field of course does not affect movements on its orthogonal hypersurfaces. Stationary models usually describe objects that rotate continuously over time like rotating planets of black holes. Static models obviously are used for non moving objects. We will now use Theorem 3.2 to prove Birkhoff's theorem:

**Theorem 3.3.** "Birkhoff's Theorem"
*In a 4-dimensional spacetime with spherically symmetric matter distribution in a vacuum (i.e. $R_{ij} = 0$) the metric must be of the form:*

$$-(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2((du^1)^2 + \sin^2(u^1)(du^2)^2).$$

*Here $M$ is the Newtonian mass and $G$ the gravity constant.*

*Proof.*
In a 4-dimensional spacetime one can choose a foliation with 2-spheres in coordinates $(v^1, v^2, u^1, u^2)$ with the metric of the 2-sphere:

$$d\tilde{s} = (du^1)^2 + \sin^2(u^1)(du^2)^2.$$

The 2-sphere is a maximally symmetric submanifold thus we can apply Theorem 3.2. The metric then is of the form:

$$\hat{g}_{v^1v^1}(v^1, v^2)(dv^1)^2 + 2\hat{g}_{v^1v^2}(dv^1dv^2) + \hat{g}_{v^2v^2}(dv^2)^2 + \hat{r}^2(v^1, v^2)d\tilde{s}^2.$$

Without loss of generality we can assume that $\hat{r}$ depends at least on $v^2$. Then we can choose new coordinates $(v^1, r)$ with the metric:

$$g_{v^1v^1}(v^1, r)(dv^1)^2 + 2g_{v^1r}dv^1dr + g_{rr}dr^2 + r^2d\tilde{s}^2.$$

Now our aim is to find coordinates which eliminate cross terms.
This means we have to find new coordinates $(t, r)$ such that the metric is of the form

$$m(t, r)dt^2 + n(t, r)dr^2 + r^2d\tilde{s}.$$

For the function $t(v^1, r)$ one has:

$$dt = \frac{\partial t}{\partial v^1}dv^1 + \frac{\partial t}{\partial r}dr$$

$$\Rightarrow dt^2 = \left(\frac{\partial t}{\partial v^1}\right)^2 (dv^1)^2 + \frac{\partial t}{\partial v^1}\frac{\partial t}{\partial r}(dv^1dr + drdv^1) + \left(\frac{\partial t}{\partial r}\right)^2 dr^2$$

$$\Rightarrow \left(\frac{\partial t}{\partial v^1}\right)^2 (dv^1)^2 = dt^2 - \frac{\partial t}{\partial v^1}\frac{\partial t}{\partial r}(dv^1 dr + dr dv^1) - \left(\frac{\partial t}{\partial r}\right)^2 dr^2.$$

Thus $m(t, r)$ and $n(t, r)$ have to fulfil the following three equations:

$$m(t,r)\left(\frac{\partial t}{\partial v^1}\right)^2 = g_{v^1 v^1},$$

$$n + m(t,r)\left(\frac{\partial t}{\partial r}\right)^2 = g_{rr},$$

$$m(t,r)\left(\frac{\partial t}{\partial v^1}\right)\left(\frac{\partial t}{\partial r}\right) = g_{v^1 r}.$$

These three functions determine $t(v^1, r)$ $m(v^1, r)$ $n(v^1, r)$ in dependence of the metric $g$.

A spacetime can be equipped with a metric of signature $(- + ++)$. Thus $m(t, r)$ or $n(t, r)$ has to be negative. At this point we assume that $m(t, r)$ is negative. Of course this is an arbitrary choice. In fact under certain circumstances it can turn out to be invalid. We can ignore this detail since it is our purpose to understand the role that Killing vector fields play in general relativity theory. One can argue though that in the tangent Minkowski space the $t$-coordinate is the time-like one. In Minkowski space the scalar product can be brought into the form: $-dt^2 + dr^2 + r^2 d\tilde{s}$. This could be seen as a motivation for our choice. The details can be found in the seventh chapter of [2]. We use the exponential function to distinguish between positive and negative terms which changes the metric to the following form:

$$-\,\mathrm{e}^{2\alpha(t,r)}\, dt^2 + \mathrm{e}^{2\beta(t,r)}\, dr^2 + r^2 d\tilde{s}^2. \tag{3.1}$$

This is the most convenient form one can get for any spherically symmetric spacetime. The next step will be to find a solution for the Einstein equations which will allow us to determine $\alpha(t, r)$ and $\beta(t, r)$.
First we have to compute the components of the Christoffel symbols, the Riemann and the Ricci tensor. To simplify notation we will write $\partial_{u^i}$ instead of $\frac{\partial}{\partial u^i}$.
The Christoffel symbols are:

$$\Gamma^t_{tt} = \partial_t \alpha, \quad \Gamma^t_{tr} = \partial_r \alpha, \quad \Gamma^t_{rr} = \mathrm{e}^{2(\beta-\alpha)}\, \partial_t \beta,$$

$$\Gamma^r_{tt} = \mathrm{e}^{2(\alpha-\beta)}\, \partial_r \alpha, \quad \Gamma^r_{tr} = \partial_t \beta, \quad \Gamma^r_{rr} = \partial_r \beta,$$

$$\Gamma^{u^1}_{ru^1} = \frac{1}{r}, \quad \Gamma^r_{u^1 u^1} = -r\,\mathrm{e}^{-2\beta}, \quad \Gamma^{u^2}_{ru^2} = \frac{1}{r},$$

$$\Gamma^r_{u^2 u^2} = -r\,\mathrm{e}^{-2\beta}\sin^2(u^1), \quad \Gamma^{u^1}_{u^2 u^2} = -\sin(u^1)\cos(u^1), \quad \Gamma^{u^2}_{u^1 u^2} = \frac{\cos(u^1)}{\sin(u^1)}.$$

The components of the Riemann tensor are:

$$R^t_{rtr} = e^{2(\beta-\alpha)}\left(\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right) + \left(\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2\right),$$

$$\begin{aligned}
R^t_{u^1tu^1} &= -r\,e^{-2\beta}\,\partial_r\alpha,\\
R^t_{u^2tu^2} &= -r\,e^{-2\beta}\sin^2(u^1)\partial_r\alpha,\\
R^t_{u^1ru^1} &= -r\,e^{-2\alpha}\,\partial_t\beta,\\
R^t_{u^2ru^2} &= -r\,e^{-2\alpha}\sin^2(u^1)\partial_t\beta,\\
R^r_{u^1ru^1} &= r\,e^{-2\beta}\,\partial_r\beta,\\
R^r_{u^2ru^2} &= r\,e^{-2\beta}\sin^2(u^1)\partial_r\beta,\\
R^{u^1}_{u^2u^1u^2} &= (1 - e^{-2\beta})\sin^2(u^1).
\end{aligned}$$

The components of the Ricci tensor are:

$$\begin{aligned}
R_{tt} &= \left(\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right) + e^{2(\alpha-\beta)}\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right),\\
R_{rr} &= -\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_1\beta\right) + e^{2(\beta-\alpha)}\left(\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right),\\
R_{tr} &= \frac{2}{r}\partial_t\beta,\\
R_{u^1u^1} &= e^{-2\beta}\left(r(\partial_r\beta - \partial_r\alpha) - 1\right) + 1,\\
R_{u^2u^2} &= R_{u^1u^1}\sin^2(u^1).
\end{aligned}$$

Now we substitute these into the Einstein equation $R_{ij} = 0$:
$R_{tr} = 0$ yields

$$\partial_t\beta = 0.$$

Since $\partial_t R_{u^1u^1} = 0$ we can use $\partial_t\beta = 0$ to get

$$\partial_t\partial_r\alpha = 0.$$

Thus we can write: $\beta = \beta(r)$ and $\alpha = f(r) + g(t)$.
Therefore the metric must be of the following form:

$$-e^{2f(r)}\,e^{2g(t)}\,dt^2 + \ldots + d\tilde{s}^2.$$

Adapting coordinates by choosing $t$ such that $g(t) = 0$ yields

$$\alpha(t, r) = f(r).$$

This means that the metric can be written as

$$-\,\mathrm{e}^{2\alpha(r)}\, dt^2 + \mathrm{e}^{2\beta(r)}\, dr^2 + r^2 d\tilde{s}^2. \tag{3.2}$$

Furthermore $R_{tt} = R_{rr} = 0$ yields

$$0 = \mathrm{e}^{2(\beta-\alpha)}\, R_{tt} + R_{rr} = \frac{2}{r}(\partial_r \alpha + \partial_t \beta)$$

$$\Rightarrow \alpha = -\beta + c\ ,\ \ c \in \mathbb{R}.$$

Again a coordinate transformation yields $c = 0$ and therefore we have

$$\alpha = -\beta.$$

Now $R_{u^1 u^1} = 0$ means

$$\mathrm{e}^{2\alpha}(2r\partial_r \alpha + 1) = 1 \Leftrightarrow \partial_r(r\, \mathrm{e}^{2\alpha}) = 1$$

$$\Rightarrow \mathrm{e}^{2\alpha} = 1 + \frac{\mu}{r},\ \ \mu \in \mathbb{R}.$$

Altogether we get:

$$-(1 - \frac{\mu}{r})dt^2 + (1 + \frac{\mu}{r})^{-1} dr^2 + r^2 d\tilde{s}^2.$$

Since no further constraints are obtained from $R_{tt} = 0$ and $R_{rr} = 0$ this is the unique solution. In order to obtain a physically viable model we set $\mu = -2GM$ where $M$ is the Newtonian mass and $G$ the gravity constant, such that we finally obtain the Schwarzschild metric

$$-(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1} dr^2 + r^2 \left((du^1)^2 + \sin^2(u^1)(du^2)^2\right).$$

□

The Schwarzschild metric is a good example for the concepts explained earlier: Equation (3.2) shows that every spherically symmetric vacuum metric has a time-like Killing vector since it does not depend on the time coordinate. Furthermore we have shown that there is an orthogonal family of hypersurfaces, namely the spheres of the foliation. Therefore the Schwarzschild metric is static. This means that we have proven that the only spherically symmetric vacuum solution for the Einstein equation is static.

## 3.2 Homogeneity and Isotropy

**Definition 3.3.** "Transitivity, Homogeneity and Isotropy"
*Let $(M, g)$ be a semi-Riemannian manifold.*

- *The **orbit** of a point $p \in M$ is the set*

$$O_p := \{q \in M \mid \exists \phi \in I(M) : q = \phi(p)\}.$$

- *$I(M)$ is called **transitive** on $M$ if $O_p = M$ for $p \in M$. The manifold $M$ is then called **homogeneous**. One should note that orbits are submanifolds since $I(M)$ is a Lie Group (see Lemma 2.3).*

- *$I(M)$ is **simply transitive** on an orbit $O_p$ if for $\phi, \Phi \in O(M)$*

$$\phi(p) = \Phi(p) \Rightarrow \phi = \Phi.$$

- *$M$ is **isotropic** about a point $p \in M$ if all isometries that leave $p$ fixed form an isometry group that is isomorphic to $SO(n)$ (the group of n-dimensional rotations).*
  *$M$ is an **isotropic manifold** if it is isotropic about all of its points.*

An important concept in general relativity theory is the **cosmological principle**. It is a hypothesis saying that at any time on a large scale the 3-dimensional space we live in is homogeneous and isotropic. In other words: On a sufficiently large scale, there are no preferred places (homogeneity) and no preferred directions (isotropy). In the cosmological principle the terms homogeneous an isotropic refer only to the spatial structure of 3-dimensional space we live in (The **perfect cosmological principle** says that the universe is homogeneous and isotropic in space and time. This hypothesis is met by the steady state models). For a rigorous mathematical formulation of the comological principle Killing vectors and isometries are crucial.

**Definition 3.4.** "Spatial Homogeneity and Isotropy"
*A 4-dimensional spacetime is spatially homogeneous (respectively isotropic) if it can be foliated by space-like hypersurfaces, such that each hypersurface is a homogeneous (respectively isotropic) manifold.*

This yields that a 4-dimensional spatially homogeneous and isotropic spacetime must have at least six Killing vectors: Three Killing vectors generating the translations that imply homogeneity on each hypersurface and three Killing vectors that generate the rotations that imply isotropy on each hypersurface.

## 3.3 Killing Vector Fields and Conservation Laws

As we know a freely falling test body moves along a geodesic $\gamma(t)$ in spacetime. The conservation Lemma 2.5 tells us that for a Killing vector field $\xi$ and a geodesic $\gamma$ on $M$ $g(\gamma', \xi)$ is constant along $\gamma$.
This means that in a coordinate chart the function

$$\sum_a \xi_a(\gamma(t))\gamma'(t)^a$$

is constant along a geodesic.
In Mechanics every Killing vector field belongs to a conservation law. In Minkowski space with its 10 Killing vector fields this would be the **four-momentum** represented by the four translations, the **angular momentum** represented by three spatial rotations and the **center-of-gravity law** represented by the Killing vector fields that belong to the special Lorentz transformations (There are semi-Riemannian manifolds that admit more conversation laws than Killing vector fields, see [18, p. 195]). The conversation property of Killing vector fields is a consequence of Noether's theorem. Basically this theorem says that a differentiable symmetry of the action of a physical system belongs to a conservation law. The action of a physical system is an integral of a Lagrangian function. Informally described: In terms of physics Noether's theorem has the consequence that time translations belong to the conservation of energy, space translations belong to the conservation of momentum and rotations belong to the conservation of angular momentum.

# Chapter 4

# Killing Vectors of Particular Lorentzian Metrics

The Gödel metric is an important solution of the Einstein field equation. It models a rotating but non-expanding universe. In their paper [16] M. Gürses, M. Plaue, M. Scherfner, T. Schönfeld and L. A. M. de Sousa have worked out an approach to introduce expansion to rotating models. They explain how to construct cosmological models by solving the differential equations arising from calculating the kinematical invariants (shear, rotation, expansion and acceleration, see Section 1.4) of an observer field in proper time description. As an application they present two generalizations of the Gödel metric. In both cases the expansion $\theta$ is non-vanishing. The expansion is defined by its **scale parameter** $s \in C^\infty(\mathbb{R})$ with $s(0) = 1$ and

$$\theta(t) = \frac{3s'(t)}{s(t)}.$$

The Gödel metric as well as the two generalized metrics have the time-like observer field $\delta_1 = \frac{\partial}{\partial t}$ and the two generalizations of the Gödel metric are then one metric with vanishing acceleration and one that is parallax-free.

In Section 4.1 the details of an approach to find the Killing vectors of the Gödel metric by T. Chrobok from his paper "Killing Vectors in Cosmological Models with Rotation" in [17, p. 106] are presented.
In Sections 4.2 and 4.3 a method to find the Killing vectors of the metrics from [16] is presented. This is done by using an approach similar to the one made by T. Chrobok to find the Killing vectors in the Gödel-type models given by Korotky and Obukhov in [17, p. 108]. Some of the calculations have been conducted with the maple® software including the "GR-Tensor" package. The source code can be found in the appendix.

## 4.1 The Gödel Metric

In the 1920s and 1930s the Friedmann-Lemaître-Robertson-Walker metric had been found as a solution of the Einstein field equation that models an expanding (or contracting) universe. It was consistent with the red shift of distant galaxies that had been observed by Hubble in the late 1920s. The metric models a spatially homogeneous, isotropic universe and has a shear-free observer field that has vanishing acceleration. The modeled universe is non-rotating though (see [18, pp. 252-268]).
In 1949 Gödel published a metric that models a rotating universe with vanishing shear and acceleration as well as vanishing expansion. The model allows time-like closed curves. Therefore it is sometimes regarded as not being very physical because this contradicts the idea of causality (In 1952 Gödel published a version where no time-like closed curves exist). Nevertheless the metric was an important contribution because it showed that Einstein's equations do not incorporate Mach's Principle. Roughly speaking Mach's Principle denies the existence of absolute space. In this chapter metrics will always be given by their representative matrices. The Gödel metric on $\mathbb{R}^4$ with coordinates $(t, x, y, z)$ and signature $(+ - --)$ is given by

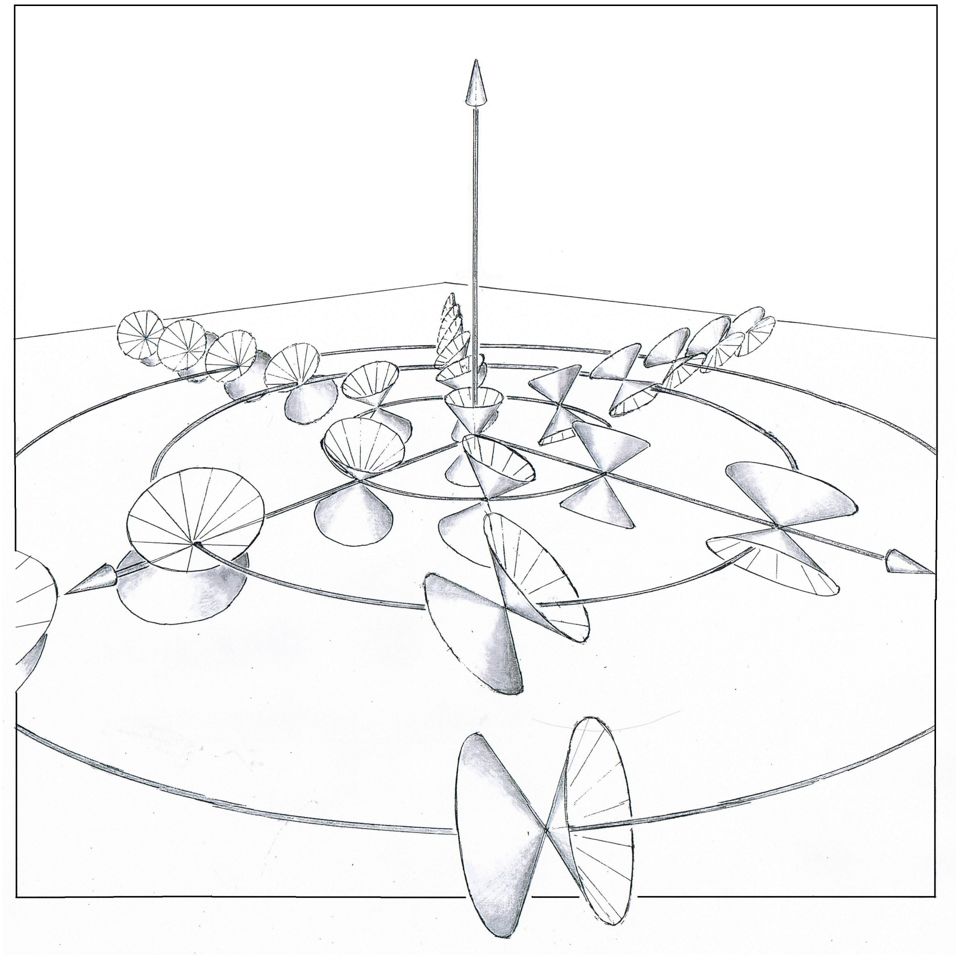
$$g = \begin{pmatrix} a^2 & 0 & a^2\,\mathrm{e}^x & 0 \\ 0 & -a^2 & 0 & 0 \\ a^2\,\mathrm{e}^x & 0 & \frac{1}{2}a^2\,\mathrm{e}^{2x} & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix}.$$

Here $a \in \mathbb{R}$ is a constant related to the rotation. The energy-momentum tensor can be interpreted as a perfect fluid. This would imply unreasonable high pressure in relation to the rest-mass density. Therefore it is often interpreted as a dust solution with cosmological constant (see [18, p. 274]). It can be shown that the Gödel metric is the only perfect fluid solution with five Killing vectors. An outline for the approach how to prove this statement can be found in [11, pp. 117-119]. The full prove is presented in [6]. In his paper [7] Gödel exhibits the following properties of his solution, where in this section $(M, g)$ denotes the Gödel spacetime:

- The metric is spatially homogeneous (as we will see later) but anisotropic.
- There exists a one-parameter group of isometries that map matter world lines to themselves.
- For every $p \in M$ there is a one-parameter group of isometries that maps $p$ to itself. This means rotational symmetry.
- A positive direction of time can be introduced.
- It is not possible to find coordinates such that the time coordinate always increases while moving in positive time direction.

- Every world line of matter is an open line of infinite length.
  For every point there exists a closed time-like curve through that point though.
- There exist no space-like 3-spaces that intersect each world line of matter at one point.
- There exists no absolute time: If there is a system of mutually exclusive 3-spaces, each of which intersects every world line of matter at one point, then there is a transformation that maps the space and its positive time direction to itself. It does not map the system of three-spaces to itself then.
- Matter rotates everywhere relative to the compass of inertia with the angular velocity $2\sqrt{\pi G \rho}$, where $\rho$ is the mean density of matter and $G$ is Newton's gravitational constant.

These are not the original statements from Gödel's paper but slightly modified versions, adapted to the notation in this work. Also some details, that are of no interest in the context of this chapter, have been left out.
One can see that the metric is the direct product of two manifolds $(M_1, g_1)$ defined by the coordinate set $(t, x, y)$ and $(M_2, g_2)$ by the coordinate $(z)$. This allows to sketch the properties of this spacetime by only considering $(M_1, g_1)$ since $(M_2, g_2)$ is just the real line. Using (generalized) cylindrical coordinates $(t, r, \phi)$ one obtains the following picture.



The sketch reveals the nature of some time-like closed curves: On the axis ($r = 0$) the light cones contain the vectors $\frac{\partial}{\partial t}$, but not the vectors $\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial r}$. From a certain radius on, the time cones have tilted and opened so much, that $\frac{\partial}{\partial \phi}$ lies inside the light cone. This means that all circles around the origin with radius greater than $\ln(1 + \sqrt{2})$ are time-like curves. A refined picture as well as more detailed explanations of the Gödel metric can be found in [8, p. 169].

In order to find the Killing vectors of the Gödel metric one has to recall the Killing equations from Remark 2.2,

$$L_\xi(g_{ij}) = \sum_{k=1}^{n} g_{ij,k}\xi^k + g_{kj}\xi^j_{,i} + g_{ik}\xi^k_{,j} = 0.$$

Using coordinates $(x^1, x^2, x^3, x^4) = (t, x, y, z)$ this yields the following system of partial differential equations:

$$\xi^1_{,1} + \mathrm{e}^x\,\xi^3_{,1} = 0, \tag{4.1}$$

$$\xi^2_{,2} = 0, \tag{4.2}$$

$$\mathrm{e}^x\,\xi^2 + 2\xi^1_{,3} + \mathrm{e}^x\,\xi^3_{,3} = 0, \tag{4.3}$$

$$\xi^4_{,4} = 0, \tag{4.4}$$

$$\xi^1_{,2} - \xi^2_{,1} + \mathrm{e}^x\,\xi^3_{,2} = 0, \tag{4.5}$$

$$\mathrm{e}^x\,\xi^2 + \mathrm{e}^x\,\xi^1_{,1} + \frac{1}{2}\,\mathrm{e}^{2x}\,\xi^3_{,1} + \xi^1_{,3} + \mathrm{e}^x\,\xi^3_{,3} = 0, \tag{4.6}$$

$$\xi^1_{,4} - \xi^4_{,1} + \mathrm{e}^x\,\xi^3_{,4} = 0, \tag{4.7}$$

$$\mathrm{e}^x\,\xi^1_{,2} + \frac{1}{2}\,\mathrm{e}^{2x}\,\xi^3_{,2} - \xi^2_{,3} = 0, \tag{4.8}$$

$$\xi^4_{,2} + \xi^2_{,4} = 0, \tag{4.9}$$

$$\mathrm{e}^x\,\xi^1_{,4} - \xi^4_{,3} + \frac{1}{2}\,\mathrm{e}^{2x}\,\xi^3_{,4} = 0, \tag{4.10}$$

First we will try to find a differential equation that characterizes $\xi^3$:
By multiplying (4.5) with $\mathrm{e}^x$ one gets:

$$\mathrm{e}^x\,\xi^1_{,2} = \mathrm{e}^x\,\xi^2_{,1} - \mathrm{e}^{2x}\,\xi^3_{,2}$$

and equation (4.8) then yields:

$$\mathrm{e}^x\,\xi^2_{,1} - \frac{1}{2}\,\mathrm{e}^{2x}\,\xi^3_{,2} - \xi^2_{,3} = 0.$$

Due to equation (4.2), after differentiating w.r.t $x$ one gets

$$\mathrm{e}^x\,\xi^2_{,1} - \mathrm{e}^{2x}\,\xi^3_{,2} - \frac{1}{2}\,\mathrm{e}^{2x}\,\xi^3_{,22} = 0 \Leftrightarrow \xi^2_{,1} - \mathrm{e}^x\,\xi^3_{,2} - \frac{1}{2}\,\mathrm{e}^x\,\xi^3_{,22} = 0.$$

After differentiating again with respect to $x$ one gets

$$
\begin{aligned}
& -\mathrm{e}^x \xi^3_{,2} - \mathrm{e}^x \xi^3_{,22} - \frac{1}{2}\mathrm{e}^x \xi^3_{,22} - \frac{1}{2}\mathrm{e}^x \xi^3_{,222} = 0 \\
\Leftrightarrow\ & \xi^3_{,2} + \frac{3}{2}\xi^3_{,22} + \frac{1}{2}\xi^3_{,222} = 0.
\end{aligned}
$$

Now we can solve this differential equation with the eigenvalue-method:

$$
\begin{pmatrix} \xi_1' \\ \xi_2' \\ \xi_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}.
$$

The eigenvalues are $\{-2, -1, 0\}$.

The corresponding eigenvectors are $\{\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$.

Therefore $\xi^3$ is of the form

$$
\xi^3 = f(t,y,z)\,\mathrm{e}^{-2x} + g(t,y,z)\,\mathrm{e}^{-x} + E(t,y,z)
$$

and we set

$$
\begin{aligned}
\xi^1 &= h(t,x,y,z), \\
\xi^2 &= j(t,y,z).
\end{aligned}
$$

Now all one has to do is to substitute these functions into the equations (4.1) - (4.10) to receive step by step all necessary informations to determine the functions $f(t,x,y,z)$, $g(t,x,y,z)$, $E(t,x,y,z)$, $h(t,x,y,z)$ and $j(t,x,y,z)$. Partial derivatives of these functions are denoted by a subscript, e.g. $h(t,x,y,z)_{xx} := \frac{\partial^2 E}{\partial^2 x}$.

Substituting these results into equation (4.8) one gets

$$
\begin{aligned}
& \mathrm{e}^x h(t,x,y,z)_x - \frac{1}{2} g(t,y,z)\,\mathrm{e}^x - f(t,y,z) - j(t,y,z)_y = 0 \\
\Rightarrow\ & h_x = \frac{1}{2} y + \mathrm{e}^{-x}(f + j_y). \qquad (4.11)
\end{aligned}
$$

By differentiating w.r.t $x$ one obtains:

$$
\begin{aligned}
& \mathrm{e}^x h(t,x,y,z)_x + \mathrm{e}^x h(t,x,y,z)_{xx} - \frac{1}{2} g(t,y,z)\,\mathrm{e}^x = 0 \\
\Leftrightarrow\ & h_x + h_{xx} - \frac{1}{2} g(t,y,z) = 0.
\end{aligned}
$$

Now one can integrate w.r.t. $x$:

$$
\begin{aligned}
&h(t,x,y,z) + h(t,x,y,z)_x - \frac{1}{2}g(t,y,z)x + c(t,y,z) = 0 \\
&\Rightarrow\; h(t,x,y,z) = -h(t,x,y,z)_x + \frac{1}{2}g(t,y,z)x + c(t,y,z).
\end{aligned}
$$

By using the formula for $h_x$ from (4.11) one can now express h(t,x,y,z) in terms of the other unknown functions:

$$
\begin{aligned}
h(t,x,y,z) = &-\frac{1}{2}g(t,y,z) - \mathrm{e}^{-x}(f(t,y,z) + j(t,y,z)_y) \\
&+ \frac{1}{2}g(t,y,z)x - c(t,y,z).
\end{aligned}
$$

Substituting this into equation (4.5) yields

$$
\begin{aligned}
&\frac{1}{2}g(t,y,z) + j(t,y,z)_t + \mathrm{e}^{-x}(f(t,y,z) - j(t,y,z)_y = 0 \\
&\Rightarrow\; g(t,y,z) = -2j(t,y,z)_t, \quad f(t,y,z) = j(t,y,z)_y.
\end{aligned} \tag{4.12}
$$

**Result:**
$\xi_1 = j(t,y,z)_t - 2\,\mathrm{e}^{-x}\,j(t,y,z)_y - j(t,y,z)_t x - c(t,y,z),$
$\xi_2 = j(t,y,z),$
$\xi_3 = j(t,y,z)_y\,\mathrm{e}^{-2x} - 2\,\mathrm{e}^{-x}\,j(t,y,z)_t + E(t,y,z).$

Now one can substitute this result into (4.1):

$$
\begin{aligned}
&j(t,y,z)_{tt} - j(t,y,z)_{yt}2\,\mathrm{e}^{-x} - j(t,y,z)_{tt}x - c(t,y,z)_t \\
&+ j(t,y,z)_{yt}\,\mathrm{e}^{-x} - 2j(t,y,z)_{tt} + E(t,y,z)_t\,\mathrm{e}^{x} = 0 \\
&\Leftrightarrow\; -j(t,y,z)_{tt}(x+1) - j(t,y,z)_{yt}\,\mathrm{e}^{-x} \\
&\qquad - c(t,y,z)_t + E(t,y,z)_t\,\mathrm{e}^{x} = 0 \\
&\Rightarrow\; j(t,y,z)_{tt} = j(t,y,z)_{yt} = E(t,y,z)_t = c(t,y,z)_t = 0 \\
&\Rightarrow\; j(t,y,z) = j_1(z)t + j_2(y,z).
\end{aligned}
$$

Note: $j_1$, $j_2$, $j_{21}$ and $j_{22}$ denote smooth functions. The index is not related to the partial derivatives of the function $j$.

**Result:**
$\xi^1 = j_1(z) - j_2(y,z)_y 2\,\mathrm{e}^{-x} - j_1(z)x - c(y,z),$
$\xi^2 = j_1(z)t + j_2(y,z),$
$\xi^3 = j_2(y,z)_y\,\mathrm{e}^{-2x} - 2j_1(z)\,\mathrm{e}^{-x} + E(y,z).$

Together with equation (4.3) this yields

$$\begin{aligned}
&j_1(z)t\,\mathrm{e}^{x} + j_2(y,z)\,\mathrm{e}^{x} + 4\,\mathrm{e}^{-x}\,j_2(y,z)_{yy}\\
&-2c(y,z)_y + j_2(y,z)_{yy}\,\mathrm{e}^{-x} + E(y,z)_y\,\mathrm{e}^{x} = 0\\
&\Leftrightarrow\ (j_1(z)t + j_2(y,z) + E(y,z)_y)\,\mathrm{e}^{x} + (5j_2(yz)_{yy})\,\mathrm{e}^{-x} - 2c(y,z)_y = 0\\
&\Rightarrow\ E(y,z)_y = -j_1(z)t - j_2(y,z),\quad j_2(y,z)_{yy} = 0,\quad c(y,z)_y = 0\\
&\Rightarrow\ j_2(y,z) = j_{21}(z)y + j_{22}(z),\quad c(y,z) = c(z).
\end{aligned}$$

**Result:**
$\xi^1 = j_1(z) - j_{21}(z)2\,\mathrm{e}^{-x} - j_1(z)x - c(z)$,
$\xi^2 = j_1(z)t + j_{21}(z)y + j_{22}(z)$,
$\xi^3 = j_{21}(z)\,\mathrm{e}^{-2x} - 2j_1(z)\,\mathrm{e}^{-x} + E(y,z)$.

By using (4.7) one gets:

$$\begin{aligned}
&j_1'(z) - j_{21}'(z)2\,\mathrm{e}^{-x} - j_1'(z)x - c'(z)\\
&-\xi^4_{,t} + j_{21}'(z)\,\mathrm{e}^{-x} - 2j_1'(z) + E(y,z)_z\,\mathrm{e}^{x} = 0\\
&\Leftrightarrow\ j_1'(z)(-x-1) - j_{21}'(z)\,\mathrm{e}^{-x} - c'(z) - \xi^4_{,t} + E(y,z)_z\,\mathrm{e}^{x} = 0\\
&\Rightarrow\ j_1(z) = j_1 \in \mathbb{R},\ j_{21}(z) = j_{21} \in \mathbb{R},\\
&\quad E(y,z) = E(y),\ \xi^4_{,t} = -c'(z).
\end{aligned}$$

**Result:**
$\xi^1 = j_1 - 2j_{21}\,\mathrm{e}^{-x} - j_1 x - c(z)$,
$\xi^2 = j_1 t + j_{21}y + j_{22}(z)$,
$\xi^3 = j_{21}\,\mathrm{e}^{-2x} - 2j_1\,\mathrm{e}^{-x} + E(y)$,
$\xi^4 = -c'(z)t + \alpha(x,y,z),\ \alpha \in C^{\infty}(\mathbb{R})$.

Substituting this result into (4.9) yields

$$\begin{aligned}
&\xi^4_{,x} = -j_{22}'(z)\\
&\Rightarrow\ \xi^4 = -c'(t)t + j_{22}'(z)x + \alpha(y,z)
\end{aligned}$$

and $\alpha(y,z)$ is restricted by (4.10):

$$\begin{aligned}
&c'(z) - \alpha(y,z)_y = 0\\
&\Rightarrow\ \alpha(y,z) = c'(z)y + \alpha(z).
\end{aligned}$$

**Result:**
$\xi^1 = j_1 - j_{21}\,\mathrm{e}^{-x} - j_1 x - c(z),$
$\xi^2 = j_1 t + j_{21} y + j_{22}(z),$
$\xi^3 = j_{21}\,\mathrm{e}^{-2x} - 2j_1\,\mathrm{e}^{-x} + E(y),$
$\xi^4 = -c'(z)t + j_{22}'(z)x + c'(z)y + \alpha(z).$

From (4.3) one now gets:

$$\begin{aligned}
&\mathrm{e}^x (j_1 t + j_{21} y + j_{22}(z)) + \mathrm{e}^x\, E'(y) = 0\\
&\Rightarrow\ E(y) = -(j_1 t + \frac{1}{2} j_{21} y + j_{22}(z))y + E \ \textit{with}\ E \in \mathbb{R}\\
&\Rightarrow\ j_1 = 0,\ \ j_{22}(z) = j_{22} \in \mathbb{R}.
\end{aligned}$$

**Result:**
$\xi^1 = -2j_{21}\,\mathrm{e}^{-x} - c(z),$
$\xi^2 = j_{21} y + j_{22},$
$\xi^3 = j_{21}\,\mathrm{e}^{-2x} - (\frac{1}{2} j_{21} y^2 + j_{22} y) + E,$
$\xi^4 = -c'(z)t + c'(z)y + \alpha(z).$

Equation (4.4) then reveals:

$$\begin{aligned}
&c''(z)(y-t) + \alpha'(z) = 0\\
&\Rightarrow\ c'' = 0,\ \alpha(z) = \alpha \in \mathbb{R}\\
&\Rightarrow\ c(z) = cz + d,\ c, d \in \mathbb{R}
\end{aligned}$$

and equation (4.10) finally yields

$$\begin{aligned}
&-c\,\mathrm{e}^x - c = 0\\
&\Rightarrow\ c = 0. \qquad (4.13)
\end{aligned}$$

**Result:**
$\xi^1 = -2j_{21}\,\mathrm{e}^{-x} + d,$
$\xi^2 = j_{21} y + j_{22},$
$\xi^3 = j_{21}\,\mathrm{e}^{-2x} - \frac{1}{2} j_{21} y^2 - j_{22} y + E,$
$\xi^4 = \alpha.$

Thus the Gödel metric has exactly five Killing vectors:

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\xi_4 = \begin{pmatrix} -2\,\mathrm{e}^{-x} \\ y \\ \mathrm{e}^{-2x} - \frac{1}{2}y^2 \\ 0 \end{pmatrix}, \ \xi_5 = \begin{pmatrix} 0 \\ 1 \\ -y \\ 0 \end{pmatrix}.$$

Note that the lower index now does not denote the covariant component but simply distinguishes the five elements of the basis given above. The structure constants can be calculated easily:

$$[\xi_1, \xi_2] = [\xi_1, \xi_3] = [\xi_2, \xi_3] = [\xi_1, \xi_4] = [\xi_1, \xi_5] = [\xi_3, \xi_4] = [\xi_3, \xi_5] = 0$$
$$\Rightarrow \ C^i_{12} = C^i_{13} = C^i_{23} = C^i_{14}$$
$$= C^i_{15} = C^i_{34} = C^i_{35} = 0 \ for \ i \in \{1, ..., 5\}.$$

$$[\xi_2, \xi_4] = dx - ydy = \xi_5$$
$$\Rightarrow \ C^5_{24} = 1 \ and \ C^i_{24} = 0 \ for \ i \neq 5.$$

$$[\xi_4, \xi_5] = -2\,\mathrm{e}^{-x}\,dt + ydx + (\mathrm{e}^{-2x}\,dt - \frac{1}{2}y^2)dy = \xi_4$$
$$\Rightarrow \ C^4_{45} = 1, \ C^i_{45} = 0 \ for \ i \neq 4.$$

$$[\xi_2, \xi_5] = -dy = -\xi_2$$
$$\Rightarrow \ C^2_{25} = -1 \ and \ C^i_{25} = 0 \ for \ i \neq 2.$$

First we will identifiy 3-dimensional Lie subalgebras: The set $\{e_1 = \delta_t, e_2 = \delta_y, e_3 = \delta_z\}$ is a **transitive**, **abelian** 3-dimensional transformation group and therefore of **Bianchi type I** acting on the hyperplanes with $x = x_0 \in \mathbb{R}$.

The set $\{e_1 = \xi_3, e_2 = \xi_2, e_3 = \xi_5\}$ with structure constants $\{\tilde{C}^k_{ij}\}_{i,j,k \in \{1,2,3\}}$ has only one non-vanishing structure constant which is $\tilde{C}^2_{23} = C^2_{25} = -1$. Using the notation from section 2.4 this yields

$$N = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

After a principle component analysis of the symmetric part and multiplication of the third vector with $\frac{1}{2}$ one obtains:

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore one has

$$n_1 = 0,\ n_2 = -1,\ n_3 = 1,\ a = 1,$$

which means that this subgroup is of **Bianchi type III** obviously acting on the hyperplanes $t = t_0$.

Less obviously as in the preceding cases it is also possible to find a subgroup of **Bianchi type VIII**.
One should note that these 3-dimensional subalgebras of Killing vector fields are tangent to hyperplanes. These hyperplanes are not space-like everywhere. Thus the Gödel spacetime can neither be spatially homogeneous nor spatially isotropic. It is a 4-dimensional homogeneous manifold though, since there exists the 4-dimensional Lie subalgebra

$$span(\xi_1, \xi_2, \xi_3, \xi_5).$$

According to the classification in section 2.4 it is of type $N0[3]$.

## 4.2 A Gödel-type Metric with Vanishing Acceleration

Define $w := \sqrt{2}\omega_0$, $\omega_0 \in \mathbb{R}$ and $s(t) \in C^\infty(\mathbb{R})$ as not constant. The constant $\omega_0$ is the rotation. The metric with vanishing acceleration from [16, p. 8] then has the representative matrix

$$g = \begin{pmatrix} 1 & 0 & \mathrm{e}^{wx} & 0 \\ 0 & -s(t)^2 & 0 & 0 \\ \mathrm{e}^{wx} & 0 & \frac{1}{2}\,\mathrm{e}^{2wx}(2-s(t)^2) & 0 \\ 0 & 0 & 0 & -s(t)^2 \end{pmatrix}.$$

$(M, g)$ denotes $\mathbb{R}^4$ with this metric $g$ as a spacetime.
Using coordinates $(x_1, x_2, x_3, x_4) = (t, x, y, z)$ the Killing equation reads:

$$\xi^1_{,1} + \mathrm{e}^{wx}\,\xi^3_{,1} = 0, \tag{4.14}$$

$$s'(t)\xi^1 + s(t)\xi^2_{,2} = 0, \tag{4.15}$$

$$-\mathrm{e}^{wx}\,s(t)s'(t)\xi^1 + w\,\mathrm{e}^{wx}(2-s(t)^2)\xi^2 + 2\xi^1_{,3} + \mathrm{e}^{wx}(2-s(t)^2)\xi^3_{,3} = 0, \tag{4.16}$$

$$s'(t)\xi^1 + s(t)\xi^4_{,4} = 0, \tag{4.17}$$

$$-s(t)^2\xi^2_{,1} + \xi^1_{,2} + \mathrm{e}^{wx}\,\xi^3_{,2} = 0, \tag{4.18}$$

$$w\,\mathrm{e}^{wx}\,\xi^2 + \mathrm{e}^{wx}\,\xi^1_{,1} + \frac{1}{2}\,\mathrm{e}^{2wx}(2-s(t)^2)\xi^3_{,1} + \xi^1_{,3} + \mathrm{e}^{wx}\,\xi^3_{,3} = 0, \tag{4.19}$$

$$-s(t)^2\xi^4_{,1} + \xi^1_{,4} + \mathrm{e}^{wx}\,\xi^3_{,4} = 0, \tag{4.20}$$

$$\mathrm{e}^{wx}\,\xi^1_{,2} + \frac{1}{2}\,\mathrm{e}^{2wx}(2-s(t))^2\xi^3_{,2} - s(t)^2\xi^2_{,3} = 0, \tag{4.21}$$

$$\xi^4_{,2} + \xi^2_{,4} = 0, \tag{4.22}$$

$$-s(t)^2\xi^4_{,3} + \mathrm{e}^{wx}\,\xi^1_{,4} + \frac{1}{2}\,\mathrm{e}^{wx}(2-s(t)^2)\xi^3_{,4} = 0. \tag{4.23}$$

We will use these equations to derive solvable differential equations for the second and fourth component of the Killing vectors:
Equation (4.20) can be written as

$$\xi^3_{,4} = \mathrm{e}^{-wx}(s(t)^2\xi^4_{,1} - \xi^1_{,4})$$

and from (4.23) one then obtains

$$\xi^4_{,3} + \frac{1}{2}\,\mathrm{e}^{wx}\,\xi^1_{,4} + \mathrm{e}^{wx}(1 - \frac{1}{2}s(t)^2)\xi^4_{,1} = 0. \tag{4.24}$$

Similarly (4.18) can be rearranged:

$$\xi^3_{,2} = \mathrm{e}^{-wx}(s(t)^2\xi^2_{,1} - \xi^1_{,2}).$$

Substituting this into (4.21) yields

$$-\xi^2_{,3} + \frac{1}{2}\,\mathrm{e}^{wx}\,\xi^1_{,2} + \mathrm{e}^{wx}(1 - \frac{1}{2}s(t)^2)\xi^2_{,1} = 0. \tag{4.25}$$

Furthermore (4.16) can be written as

$$\xi^2 = \frac{s(t)s'(t)}{w(2 - s(t)^2)}\xi^1 - \frac{2}{w(2 - s(t)^2)}\,\mathrm{e}^{-wx}\,\xi^1_{,3} - \frac{1}{w}\xi^3_{,3}.$$

After substituting this into (4.19) one obtains

$$\mathrm{e}^{wx}\,\frac{s(t)s'(t)}{2 - s(t)^2}\xi^1 + (1 - \frac{2}{2 - s(t)^2})\xi^1_{,3} + \mathrm{e}^{wx}\,\xi^1_{,1} + \mathrm{e}^{2wx}(1 - \frac{1}{2}s(t)^2)\xi^3_{,1} = 0.$$

Since (4.14) can be read as $\xi^3_{,1} = -\,\mathrm{e}^{-wx}\,\xi^1_{,1}$,
it can be easily substituted into the preceding equation which yields

$$\frac{s(t)s'(t)}{2 - s(t)^2}\xi_1 + (1 - \frac{2}{2 - s(t)^2})\,\mathrm{e}^{-wx}\,\xi^1_{,3} + \frac{1}{2}s(t)^2\xi^1_{,1} = 0. \tag{4.26}$$

Now one can rearrange (4.15):

$$\xi^1 = -\frac{s(t)}{s'(t)}\xi^2_{,2}$$

and substitute it into (4.17):

$$\xi^4_{,4} - \xi^2_{,2} = 0. \tag{4.27}$$

This yields:

$$\xi^1 = -\frac{s(t)}{s'(t)}\xi^2_{,2} = -\frac{s(t)}{s'(t)}\xi^4_{,4}. \tag{4.28}$$

At this point one can use the equations (4.24) - (4.28) to find a system of differential equations to determine $\xi^2$ and $\xi^4$. The first equation in this system is (4.27):

$$\boxed{\xi^4_{,4} - \xi^2_{,2} = 0.} \tag{4.29}$$

The second one is obtained by substituting (4.28) into (4.24):

$$\boxed{-\xi^4_{,3} - \mathrm{e}^{wx}\,\frac{1}{2}\frac{s(t)}{s'(t)}\xi^4_{,44} + \mathrm{e}^{wx}(1 - \frac{1}{2}s(t)^2)\xi^4_{,1} = 0.} \tag{4.30}$$

Similarly one can substitute (4.28) into (4.25):

$$\boxed{-\xi^2_{,3} - \mathrm{e}^{wx}\frac{s(t)}{s'(t)}\xi^2_{,22} + \mathrm{e}^{wx}(1 - \frac{1}{2}s(t)^2)\xi^2_{,1} = 0.} \tag{4.31}$$

By substituting (4.28) into (4.26) one gets:

$$\boxed{-\frac{s(t)^2}{2 - s(t)^2}\xi^2_{,2} - (1 - \frac{2}{2 - s(t)^2})\,\mathrm{e}^{-wx}\frac{s(t)}{s'(t)}\xi^2_{,23} - \frac{1}{2}\frac{s(t)^3}{s'(t)}\xi^2_{,21} = 0,} \tag{4.32}$$

$$\boxed{-\frac{s(t)^2}{2 - s(t)^2}\xi^4_{,4} - (1 - \frac{2}{2 - s(t)^2})\,\mathrm{e}^{-wx}\frac{s(t)}{s'(t)}\xi^4_{,43} - \frac{1}{2}\frac{s(t)^3}{s'(t)}\xi^4_{,41} = 0.} \tag{4.33}$$

The last equation is (4.22) from the original system:

$$\boxed{\xi^4_{,2} + \xi^2_{,4} = 0.} \tag{4.34}$$

Now one can differentiate (4.29) w.r.t. $z$ and add (4.34) differentiated w.r.t. $x$:

$$\xi^4_{,22} + \xi^4_{,44} = 0.$$

The other way round one gets

$$\xi^2_{,22} + \xi^2_{,44} = 0.$$

These are the **Laplace equations** for $\xi_2$ and $\xi_4$ w.r.t. the variables $x$ and $z$. Therefore $\xi^2$ and $\xi^4$ can be chosen as

$$\begin{aligned}\xi^2 =\ & a(t,y) + xb(t,y) + (x^2 - z^2)c(t,y) + xzd(t,y) + zE(t,y)\\ & + (xz^3 - zx^3)f(t,y) + \ln(\sqrt{x^2+z^2})g(t,y) \pm h(t,y)\,\mathrm{e}^{k(t,y)(z+ix)},\\ \xi^4 =\ & l(t,y) + xn(t,y) + (x^2 - z^2)o(t,y) + xzp(t,y) + zq(t,y)\\ & + (xz^3 - zx^3)r(t,y) + \ln(\sqrt{x^2+z^2})S(t,y) \pm u(t,y)\,\mathrm{e}^{v(t,y)(z+ix)}\end{aligned}$$

with undetermined functions $a(t,y)$, $b(t,y)$, $c(t,y)$, $d(t,y)$, $E(t,y)$, $f(t,y)$, $g(t,y)$, $h(t,y)$, $k(t,y)$, $l(t,y)$, $n(t,y)$, $o(t,y)$, $p(t,y)$, $q(t,y)$, $r(t,y)$, $S(t,y)$, $u(t,y)$, $v(t,y)$.
The idea behind this choice is that one uses a linear combination of simple functions that solve the Laplace equation with respect to $\xi^2$ and $\xi^4$ in the variables $(x,z)$. Of course there exist more solutions to the Laplace equation,

but as we will see later this one suffices for our purpose.

Substituting this into (4.29) yields:

$$
\begin{aligned}
&- z(2o(t,y) + d(t,y)) + x(p(t,y) - 2c(t,y)) + (g(t,y) - b(t,y)) \\
&+ (3xz^2 - x^3)r(t,y) + S(t,y)\frac{z}{x^2+z^2} - (z^3 - 3zx^2)f(t,y) \\
&+ \frac{x}{x^2+z^2}g(t,y) \pm v(t,y)u(t,y)\,\mathrm{e}^{\pm v(t,y)(z+ix)} \\
&\pm k(t,y)h(t,y)i\,\mathrm{e}^{k(t,y)(z+ix)} = 0 \\
&\Rightarrow\ 2o(t,y) = -d(t,y),\ p(t,y) = 2c(t,y),\ q(t,y) = b(t,y), \\
&\qquad r(t,y) = S(t,y) = k(t,y) = f(t,y) = g(t,y) = v(t,y) = 0.
\end{aligned}
$$

Additionally from (4.34) one obtains:

$$n(t,y) = -E(t,y).$$

**Result:**

$\xi^2 = a(t,y) + xb(t,y) + (x^2 - z^2)c(t,y) + xzd(t,y) + zE(t,y)$,

$\xi^4 = l(t,y) - xE(t,y) - (x^2 - z^2)\frac{1}{2}d(t,y) + xz2c(t,y) + zb(t,y)$.

The next step is that one substitutes this result into (4.30):

$$
\begin{aligned}
&- l(t,y)_y + xE(t,y)_y + (x^2 - z^2)\frac{1}{2}d(t,y)_y - xz2c(t,y)_y - zb(t,y)_y \\
&- \frac{1}{2}\,\mathrm{e}^{wx}\,\frac{s(t)}{s'(t)}d(t,y) + \mathrm{e}^{wx}(1 - \frac{1}{2}s(t)^2)(l(t,y)_t - xE(t,y)_t \\
&- (x^2 - z^2)\frac{1}{2}d(t,y)_t + xz2c(t,y)_t + zb(t,y)_t) = 0 \\
&\Rightarrow\ \frac{s(t)}{s'(t)}d = (2 - s(t)^2)l'(t), \\
&\qquad l(t,y) = l(t),\ E(t,y) = E \in \mathbb{R},\ d(t,y) = d \in \mathbb{R}, \\
&\qquad c(t,y) = c \in \mathbb{R},\ b(t,y) = b \in \mathbb{R} \\
&\Rightarrow\ l'(t,y) = \frac{s(t)}{s'(t)(2 - s(t)^2)}d \text{ with } d \in \mathbb{R}.
\end{aligned}
$$

Similarly one obtains from (4.31):

$$
\begin{aligned}
&- a(t,y)_y - \frac{1}{2}\,\mathrm{e}^{wx}\,\frac{s(t)}{s'(t)}2c + \mathrm{e}^{wx}(1 - \frac{1}{2}s(t)^2)a(t,y)_t = 0 \\
&\Rightarrow\ a(t,y) = a(t),\quad \frac{s(t)}{s'(t)}c = (1 - \frac{s(t)^2}{2})a'(t) \\
&\Rightarrow\ a'(t) = \frac{s(t)}{s'(t)(1 - \frac{1}{2}s(t)^2)}c.
\end{aligned}
$$

**Result:**
$\xi^2 = a(t) + bx + (x^2 - z^2)c + xzd + Ez,$
$\xi^4 = l(t) - Ex - \frac{1}{2}(x^2 - z^2)d + 2cxz + bz.$

Now (4.32) reads as

$$-\frac{s(t)^2}{2 - s(t)^2}(b + 2xc + zd) = 0$$
$$\Rightarrow \; b = c = d = 0.$$

**Result:**
$\xi^2 = a(t) + Ez,$
$\xi^4 = l(t) - Ex.$

For the functions $a(t)$ and $l(t)$ (4.30) and (4.31) yield

$$l'(t) = a'(t) = 0.$$

**Result:**
$\xi^2 = a + Ez,$
$\xi^4 = l - Ex.$

$\xi^1$ is now determined by (4.15):

$$\xi^1 = -\frac{-s'(t)}{s(t)}\xi^2_{,2} = 0$$
$$\Rightarrow \; \xi^1 = 0.$$

Furthermore (4.14) yields:

$$\xi^3(t, x, y, z) = \xi^3(x, y, z)$$

and (4.18) now reads as

$$\xi^3_{,2} = 0$$
$$\Rightarrow \; \xi^3(x, y, z) = \xi^3(y, z).$$

From (4.16) one obtains

$$\xi^3_{,3} = -(Ez + a)w$$
$$\Rightarrow \ \xi^3 = (Ex + a)wy + C(z).$$

Finally (4.20) yields

$$- wEy + C'(z) = 0$$
$$\Rightarrow \ E = 0, \ C(z) = C \in \mathbb{R}.$$

**Result:**
$\xi^1 = 0,$
$\xi^2 = a,$
$\xi^3 = -way + E,$
$\xi_4 = l.$

Therefore this metric has three Killing vectors:

$$\begin{pmatrix} 0 \\ 1 \\ -\sqrt{2}\omega_0 y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The number of Killing vectors obviously depends on the function $s(t)$. We have found three linearly independent solutions that do not depend on $s(t)$. Now we will examine a few functions that could take the place of $s(t)$ and show that in these cases the Killing vectors we have found already are the only ones. This is done by using Theorem 2.2. We will prove for some $p \in M$ that the system of linear equations $L_{\xi_p} R = 0$ contains seven linearly independent equations. In each case this is done by presenting two $7 \times 7$-minors which have only one zero in common: $\omega_0 = 0$ (The system has $\omega_0$ as a parameter). In each case we will have to consider the representative matrix of the linear system

$$(L_{\xi_p} R_{abcd} \ a, b, c, d \in \{1, ..., 4\})$$

at a point $p \in \mathbb{R}^4$. For the sake of simplicity in each case only the following results will be presented:

1. A point $p \in M$,

2. two sets of components of $R$,

$$\{R_{a_m b_m c_m d_m}\}_{m\in\{1,...,7\}},\ \{R_{i_n j_n k_n l_n}\}_{n\in\{1,...,7\}},$$

3. the representative matrices of the linear systems

$$(L_{\xi_p} R_{a_m b_m c_m d_m} = 0,\ m \in \{1,...,7\}),\ (L_{\xi_p} R_{i_n j_n k_n l_n} = 0,\ n \in \{1,...,7\})$$

with respect to the variables

$$\xi^1,\ \xi^2,\ \xi^3,\ \xi^4,\ \xi^1_{;2},\ \xi^2_{;4},\ \xi^3_{;4},$$

4. the determinants of these matrices (minors) and their zeros.

Furthermore the assumption $s(0) = 1$ will be dropped here, since it can be achieved by a simple translation.

**Case 1:** $s(t) = t^3$

This case implies for the expansion:

$$\theta(t) = \frac{9}{t}.$$

Furthermore the metric has a singularity in this case.
We choose: $p = (0, 0, 0, 0)$.

First set of equations:

$$L_{\xi_p} R_{1212} = 0,$$
$$L_{\xi_p} R_{1213} = 0,$$
$$L_{\xi_p} R_{1214} = 0,$$
$$L_{\xi_p} R_{1223} = 0,$$
$$L_{\xi_p} R_{1224} = 0,$$
$$L_{\xi_p} R_{1234} = 0,$$
$$L_{\xi_p} R_{1313} = 0.$$

The representative matrix of these equations w.r.t the variables given above is:

$$\begin{pmatrix} -16 + \frac{28}{9}\omega_0^2 & -2\sqrt{2}\omega_0 & -4 + \frac{40}{9}\omega_0^2 & 0 & -\frac{8}{3}\sqrt{2}\omega_0 & 0 & 0 \\ -7\sqrt{2}\omega_0 & \frac{10}{3}\omega_0^2 + 9 & -11\omega_0\sqrt{2} & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -\omega_0\sqrt{2} & 0 & \frac{1}{3}\omega_0^2 & -\frac{4}{3}\omega_0\sqrt{2} \\ -6 - \frac{8}{9}\omega_0^2 & \left(\frac{39}{6} - \frac{2}{6}\omega_0^2\right)\omega_0\sqrt{2} & 21 - \frac{26}{9}\omega_0^2 & 0 & \frac{8}{3}\omega_0\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \omega_0^2 + 9 & 0 & -\omega_0\sqrt{2} & 6 \\ 0 & 0 & 0 & -\frac{1}{2}\omega_0\sqrt{2} & 0 & -\frac{1}{3}\omega_0^2 & \frac{4}{3}\omega_0\sqrt{2} \\ -22 + \frac{10}{3}\omega_0^2 & -18\omega_0\sqrt{2} & 44 - \frac{8}{3}\omega_0^2 & 0 & -6\omega_0\sqrt{2} & 0 & 0 \end{pmatrix}.$$

The determinant of this matrix is

$$32832\omega_0^6 + \frac{10592}{3}\omega_0^8 - \frac{22144}{81}\omega_0^{10}.$$

The zeros of the determinant are

$$\left\{0, \pm\frac{3}{692}\sqrt{343578 + 1038\sqrt{425113}}\right\}.$$

Second set of equations:

$$
\begin{aligned}
&L_{\xi_p} R_{1212} = 0, \\
&L_{\xi_p} R_{1213} = 0, \\
&L_{\xi_p} R_{1214} = 0, \\
&L_{\xi_p} R_{1223} = 0, \\
&L_{\xi_p} R_{1224} = 0, \\
&L_{\xi_p} R_{1234} = 0, \\
&L_{\xi_p} R_{1414} = 0.
\end{aligned}
$$

The representative matrix of these equations w.r.t the variables given above is:

$$
\begin{pmatrix}
-16+\frac{28}{9}\omega_0^2 & -2\sqrt{2}\omega_0 & -4+\frac{40}{9}\omega_0^2 & 0 & -\frac{8}{3}\sqrt{2}\omega_0 & 0 & 0 \\
-7\sqrt{2}\omega_0 & \frac{10}{3}\omega_0^2+9 & -11\sqrt{2}\omega_0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{2}\omega_0 & 0 & \frac{1}{3}\omega_0^2 & -\frac{4}{3}\sqrt{2}\omega_0 \\
-6-\frac{8}{9}\omega_0^2 & \left(+\frac{39}{6}-\frac{2}{6}\omega_0^2\right)\omega_0\sqrt{2} & 21-\frac{26}{9}\omega_0^2 & 0 & \frac{8}{3}\omega_0 & 0 & 0 \\
0 & 0 & 0 & \omega_0^2+9 & 0 & -\sqrt{2}\omega_0 & 6 \\
0 & 0 & 0 & -\frac{1}{2}\sqrt{2}\omega_0 & 0 & -\frac{1}{3}\omega_0^2 & \frac{4}{3}\sqrt{2}\omega_0 \\
-16+\frac{4}{3}\omega_0^2 & 0 & -4+\frac{4}{3}\omega_0^2 & 0 & -2\sqrt{2}\omega_0 & 0 & 0
\end{pmatrix}.
$$

The determinant of this matrix is

$$\frac{9728}{9}\omega_0^8 - \frac{3584}{81}\omega_0^{10}.$$

The zeros of the determinant are:

$$\{\pm\frac{3}{7}\sqrt{133}, 0\}.$$

**Case 2:** $s(t) = \mathrm{e}^t$

This is the case with constant expansion. We choose: $p = (0, 0, 0, 0)$.
First set of equations:

$$
\begin{aligned}
&L_{\xi_p} R_{1212} = 0, \\
&L_{\xi_p} R_{1213} = 0, \\
&L_{\xi_p} R_{1214} = 0, \\
&L_{\xi_p} R_{1223} = 0, \\
&L_{\xi_p} R_{1224} = 0, \\
&L_{\xi_p} R_{1424} = 0, \\
&L_{\xi_p} R_{1313} = 0.
\end{aligned}
$$

The representative matrix of these equations w.r.t the variables given above is:

$$
\begin{pmatrix}
\frac{-16}{27} + \frac{28}{27}\omega_0^2 & 0 & -\frac{16}{27} + \frac{40}{27}\omega_0^2 & 0 & -\frac{8}{9}\sqrt{2}\omega_0 & 0 & 0 \\
-\frac{22}{27}\sqrt{2}\omega_0 & \frac{10}{9}\omega_0^2 + \frac{4}{9} & -\frac{46}{27}\sqrt{2}\omega_0 & 0 & \frac{4}{9} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{9}\sqrt{2}\omega_0 & 0 & \frac{1}{3}\omega_0^2 & -\frac{4}{9}\sqrt{2}\omega_0 \\
-\frac{8}{27} - \frac{8}{27}\omega_0^2 & \sqrt{2}\omega_0\left(\frac{-3}{9}\omega_0^2 + \frac{6}{9}\right) & \frac{28}{27} - \frac{26}{27}\omega_0^2 & 0 & \frac{8}{9}\sqrt{2}\omega_0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{9} + \frac{1}{3}\omega_0^2 & 0 & -\frac{1}{3}\sqrt{2}\omega_0 & \frac{8}{9} \\
0 & 0 & 0 & \frac{11}{18}\sqrt{2}\omega_0 & 0 & 0 & \frac{7}{9}\sqrt{2}\omega_0 \\
-\frac{16}{27} + \frac{4}{9}\omega_0^2 & \frac{2}{9}\sqrt{2}\omega_0 & -\frac{16}{27} + \frac{4}{9}\omega_0^2 & 0 & -\frac{2}{3}\sqrt{2}\omega_0 & 0 & 0
\end{pmatrix}.
$$

The determinant of this matrix is

$$\frac{-82432}{531441}\omega_0^{10} + \frac{33152}{531441}\omega_0^{12}.$$

The zeros of this determinant are:

$$\{\pm\frac{2}{37}\sqrt{851}, 0\}.$$

In the case $s(t) = \mathrm{e}^{at}$ the representative matrix has the same vanishing components and the other entries look slightly different. In this case one obtains the determinant

$$\frac{-82432}{531441}a^2\omega_0^{10} + \frac{33152}{531441}a^6\omega_0^{12}$$

which has the zeros

$$\left\{0, \pm\frac{2}{37}\sqrt{851}\, a\right\}.$$

Second set of equations:

$$
\begin{aligned}
&L_{\xi_p} R_{1212} = 0, \\
&L_{\xi_p} R_{1213} = 0, \\
&L_{\xi_p} R_{1214} = 0, \\
&L_{\xi_p} R_{1223} = 0, \\
&L_{\xi_p} R_{1224} = 0, \\
&L_{\xi_p} R_{1314} = 0, \\
&L_{\xi_p} R_{1424} = 0.
\end{aligned}
$$

The representative matrix of these equations w.r.t the variables given above is:

$$
\begin{pmatrix}
-\frac{16}{27} + \frac{28}{27}\omega_0^2 & 0 & -\frac{16}{27} + \frac{40}{27}\omega_0^2 & 0 & -\frac{8}{9}\sqrt{2}\omega_0 & 0 & 0 \\
-\frac{22}{27}\sqrt{2}\omega_0 & \frac{10}{9}\omega_0^2 + \frac{4}{9} & -\frac{46}{27}\sqrt{2}\omega_0 & 0 & \frac{4}{9} & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{9}\sqrt{2}\omega_0 & 0 & \frac{1}{3}\omega_0^2 & -\frac{4}{9}\sqrt{2}\omega_0 \\
\frac{-8}{27} - \frac{8}{27}\omega_0^2 & \sqrt{2}\omega_0\left(\frac{-3}{9}\omega_0^2 + \frac{6}{9}\right) & \frac{28}{27} - \frac{26}{27}\omega_0^2 & 0 & \frac{8}{9}\sqrt{2}\omega_0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{9} + \frac{1}{3}\omega_0^2 & 0 & -\frac{1}{3}\sqrt{2}\omega_0 & \frac{8}{9} \\
0 & 0 & 0 & \frac{4}{9} + \frac{1}{3}\omega_0^2 & 0 & \frac{-2}{3}\sqrt{2}\omega 0 & \frac{16}{9} + \frac{1}{3}\omega_0^2 \\
\frac{10}{27}\sqrt{2}\omega_0 & \frac{-2}{9} - \frac{1}{3}\omega_0^2 & \frac{19}{27}\sqrt{2} & 0 & \frac{-2}{9} & 0 & 0
\end{pmatrix}.
$$

The determinant of this matrix is

$$\frac{2560}{177147}\omega_0^{10} - \frac{896}{177147}\omega_0^{12}.$$

The zeros of the determinant are

$$\{\pm\frac{2}{7}\sqrt{35}, 0\}.$$

In the case $s(t) = \mathrm{e}^{at}$ the representative matrix has the same vanishing components and the other entries look slightly different. In this case one obtains the determinant

$$\frac{2560}{177147}a^8\omega_0^{10} - \frac{896}{177147}a^6\omega_0^{12}$$

which has the zeros

$$\left\{0, \pm\frac{2}{7}\sqrt{35}\ a\right\}.$$

**Case 3:** $s(t) = \sin(t)$

In this case the expansion is of the form

$$\theta(t) = \cot(t)$$

and again this is an example for a metric that has a singularity. We choose $p = (\frac{\pi}{4}, 0, 0, 0)$.
First set of equations:

$$\begin{aligned} L_{\xi_p} R_{1212} &= 0, \\ L_{\xi_p} R_{1213} &= 0, \\ L_{\xi_p} R_{1214} &= 0, \\ L_{\xi_p} R_{1223} &= 0, \\ L_{\xi_p} R_{1224} &= 0, \\ L_{\xi_p} R_{1234} &= 0, \\ L_{\xi_p} R_{1414} &= 0. \end{aligned}$$

The representative matrix of these equations w.r.t the variables given above is:

$$\begin{pmatrix} \frac{296}{343}\omega_0^2 + \frac{24}{343} & \frac{-8}{7}\sqrt{2}\omega_0 & \frac{24}{343} + \frac{352}{343}\omega_0^2 & 0 & -\frac{32}{49}\sqrt{2}\omega_0 & 0 & 0 \\ -\frac{176}{343}\sqrt{2}\omega_0 & \frac{-6}{49} + \frac{88}{49}\omega_0^2 & \frac{496}{343}\sqrt{2}\omega_0 & 0 & \frac{-6}{49} & 0 & 0 \\ 0 & 0 & 0 & \frac{-2}{49}\sqrt{2}\omega_0 & 0 & \frac{4}{7}\omega_0^2 & \frac{-16}{49}\sqrt{2}\omega_0 \\ \frac{18}{343} - \frac{82}{343}\omega_0^2 & \left(1 - \frac{6}{7}\omega_0^2\right)\sqrt{2}\omega_0 & \frac{-45}{343} - \frac{334}{343}\omega_0^2 & 0 & \frac{40}{49}\sqrt{2}\omega_0 & 0 & 0 \\ 0 & 0 & 0 & \frac{23}{49} + \frac{2}{7}\omega_0^2 & 0 & \frac{-2}{7}\sqrt{2}\omega_0 & \frac{-12}{49} \\ 0 & 0 & 0 & \frac{-3}{98}\sqrt{2}\omega_0 & 0 & \frac{-4}{7}\omega_0^2 & \frac{16}{49}\sqrt{2}\omega_0 \\ \frac{24}{343} + \frac{16}{49}\omega_0^2 & \frac{-52}{49}\sqrt{2}\omega_0 & \frac{24}{343} + \frac{16}{49}\omega_0^2 & 0 & \frac{-4}{7}\sqrt{2}\omega_0 & 0 & 0 \end{pmatrix}.$$

The determinant of this matrix is

$$\frac{884736}{40353607}\omega_0^8 + \frac{57212928}{1977326743}\omega_0^{10}.$$

This determinant obviously has no real zeros (only complex ones).

## 4.3 A Parallax-free Gödel-type Metric

Define $\alpha(t,x) := \sqrt{2}\omega_0 x + \frac{1}{3}\theta t$ with $\omega_0, \theta \in \mathbb{R}$. In this case the scale parameter is defined as

$$s(t) = \mathrm{e}^{\frac{\theta t}{3}}$$

such that the constant $\theta$ is the expansion. The constant $\omega_0$ is the rotation. The parallax-free metric from [16, p. 10] then has the representative matrix

$$g = \begin{pmatrix} 1 & 0 & \mathrm{e}^{\sqrt{2}\omega_0 x + \frac{1}{3}\theta t} & 0 \\ 0 & -\mathrm{e}^{\frac{2}{3}\theta t} & 0 & 0 \\ \mathrm{e}^{\sqrt{2}\omega_0 x + \frac{1}{3}\theta t} & 0 & \frac{1}{2}\mathrm{e}^{2\sqrt{2}\omega_0 x + \frac{2}{3}\theta t} & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{\frac{2}{3}\theta t} \end{pmatrix}.$$

$(M, g)$ denotes $\mathbb{R}^4$ with this metric $g$ as a spacetime.
Using coordinates $(x_1, x_2, x_3, x_4) = (t, x, y, z)$ the Killing equation reads:

$$\xi^1_{,1} + \mathrm{e}^{\alpha(t,x)}\,\xi^3_{,1} = 0, \tag{4.35}$$

$$\frac{1}{3}\theta\xi^1 + \xi^2_{,2} = 0, \tag{4.36}$$

$$\frac{1}{3}\theta\,\mathrm{e}^{2\alpha(t,x)}\,\xi^1 + \sqrt{2}\omega_0\,\mathrm{e}^{2\alpha(t,x)}\,\xi^2 + 2\,\mathrm{e}^{\alpha(t,x)}\,\xi^1_{,3} + \mathrm{e}^{2\alpha(t,x)}\,\xi^3_{,3} = 0, \tag{4.37}$$

$$\frac{1}{3}\theta\xi^1 + \xi^4_{,4} = 0, \tag{4.38}$$

$$-\mathrm{e}^{\frac{2}{3}\theta t}\,\xi^2_{,1} + \xi^1_{,2} + \mathrm{e}^{\alpha(t,x)}\,\xi^3_{,2} = 0, \tag{4.39}$$

$$\frac{1}{3}\theta\xi^1 + \sqrt{2}\omega_0\xi^2 + \xi^1_{,1} + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)}\,\xi^3_{,1} + \mathrm{e}^{-\alpha(t,x)}\,\xi^1_{,3} + \xi^3_{,3} = 0, \tag{4.40}$$

$$-\mathrm{e}^{\frac{2}{3}\theta t}\,\xi^4_{,1} + \xi^1_{,4} + \mathrm{e}^{\alpha(t,x)}\,\xi^3_{,4} = 0, \tag{4.41}$$

$$\mathrm{e}^{\alpha(t,x)}\,\xi^1_{,2} + \frac{1}{2}\,\mathrm{e}^{2\alpha(t,x)}\,\xi^3_{,2} - \mathrm{e}^{\frac{2}{3}\theta t}\,\xi^2_{,3} = 0, \tag{4.42}$$

$$\xi^4_{,2} + \xi^2_{,4} = 0, \tag{4.43}$$

$$-\mathrm{e}^{\frac{2}{3}\theta t}\,\xi^4_{,3} + \mathrm{e}^{\alpha(t,x)}\,\xi^1_{,4} + \frac{1}{2}\,\mathrm{e}^{2\alpha(t,x)}\,\xi^3_{,4} = 0. \tag{4.44}$$

Again we will use these equations to derive a solvable differential equation for some of the components of the wanted Killing vectors:

Equation (4.41) can be written as

$$\xi^3_{,4} = \mathrm{e}^{-\alpha(t,x) + \frac{2}{3}\theta t}\,\xi^4_{,1} - \mathrm{e}^{-\alpha(t,x)}\,\xi^1_{,4}$$

and from (4.44) one obtains

$$-\mathrm{e}^{\frac{2}{3}\theta t}\,\xi^4_{,3} + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)+\frac{2}{3}\theta t}\,\xi^4_{,1} + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)}\,\xi^1_{,4} = 0. \tag{4.45}$$

Similarly (4.39) can be rearranged:

$$\xi^3_{,2} = \mathrm{e}^{-\alpha(t,x)+\frac{2}{3}\theta t}\,\xi^2_{,1} - \mathrm{e}^{-\alpha(t,x)}\,\xi^1_{,2}.$$

Substituting this into (4.42) yields

$$-\mathrm{e}^{\frac{2}{3}\theta t}\,\xi^2_{,3} + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)+\frac{2}{3}\theta t}\,\xi^2_{,1} + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)}\,\xi^1_{,2} = 0. \tag{4.46}$$

Furthermore (4.37) can be written as

$$\frac{1}{3}\theta\,\mathrm{e}^{\alpha(t,x)}\,\xi^1 + \sqrt{2}\omega_0\,\mathrm{e}^{\alpha(t,x)}\,\xi^2 = -2\xi^1_{,3} - \mathrm{e}^{\alpha(t,x)}\,\xi^3_{,3}.$$

After substituting this into (4.40) one obtains

$$-\xi^1_{,3} + \mathrm{e}^{\alpha(t,x)}\,\xi^1_{,1} + \frac{1}{2}\,\mathrm{e}^{2\alpha(t,x)}\,\xi^3_{,1} = 0.$$

Since (4.35) can be read as $\xi^3_{,1} = -\,\mathrm{e}^{-\alpha(t,x)}\,\xi^1_{,1}$,
it can easily be substituted into the preceding equation which yields

$$-\xi^1_{,3} + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)}\,\xi^1_{,1} = 0. \tag{4.47}$$

Now one can rearrange (4.36):

$$\xi^1 = -\frac{3}{\theta}\xi^2_{,2},$$

and substitute it into (4.38):

$$\xi^2_{,2} - \xi^4_{,4} = 0. \tag{4.48}$$

This yields:

$$\xi^1 = -\frac{3}{\theta}\xi^2_{,2} = -\frac{3}{\theta}\xi^4_{,4}. \tag{4.49}$$

At this point one can use the equations (4.45) - (4.49) to find a system of differential equations to determine $\xi^2$ and $\xi^4$.
The first equation in this system is (4.48):

$$\boxed{\xi^2_{,2} - \xi^4_{,4} = 0.} \tag{4.50}$$

The second one is obtained by substituting (4.49) into (4.45):

$$\boxed{-\,\mathrm{e}^{\frac{2}{3}\theta t}\,\xi^4_{,3} + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)+\frac{2}{3}\theta t}\,\xi^4_{,1} - \frac{3}{2\theta}\,\mathrm{e}^{\alpha(t,x)}\,\xi^4_{,44} = 0.} \tag{4.51}$$

Similarly one can substitute (4.49) into (4.46):

$$\boxed{-\,\mathrm{e}^{\frac{2}{3}\theta t}\,\xi^2_{,3} + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)+\frac{2}{3}\theta t}\,\xi^2_{,1} - \frac{3}{2\theta}\,\mathrm{e}^{\alpha(t,x)}\,\xi^2_{,22} = 0.} \tag{4.52}$$

Substituting (4.49) into (4.47) yields:

$$\boxed{\xi^2_{,23} - \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)}\,\xi^2_{,21} = 0,} \tag{4.53}$$

$$\boxed{\xi^4_{,43} - \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)}\,\xi^4_{,41} = 0.} \tag{4.54}$$

The last equation is (4.43):

$$\boxed{\xi^4_{,2} + \xi^2_{,4} = 0.} \tag{4.55}$$

Now one can differentiate (4.50) w.r.t. $z$ and add (4.55) differentiated w.r.t. $x$:

$$\xi^4_{,22} + \xi^4_{,44} = 0.$$

The other way round one gets

$$\xi^2_{,22} + \xi^2_{,44} = 0.$$

These are the Laplace equations for $\xi_2$ and $\xi_4$ w.r.t. the variables $x$ and $z$. Therefore just as in the last section $\xi^2$ and $\xi^4$ can be chosen as

$$\begin{aligned}
\xi^2 =\; & a(t,y) + xb(t,y) + (x^2 - z^2)c(t,y) + xzd(t,y) + zE(t,y) \\
& + (xz^3 - zx^3)f(t,y) + \ln(\sqrt{x^2+z^2})g(t,y) \pm h(t,y)\,\mathrm{e}^{k(t,y)(z+ix)}, \\
\xi^4 =\; & l(t,y) + xn(t,y) + (x^2 - z^2)o(t,y) + xzp(t,y) + zq(t,y), \\
& + (xz^3 - zx^3)r(t,y) + \ln(\sqrt{x^2+z^2})S(t,y) \pm u(t,y)\,\mathrm{e}^{v(t,y)(z+ix)}\,.
\end{aligned}$$

Just as in the last section after applying (4.50) and (4.55) to these functions one obtains

$$\xi^2 = a(t,y) + xb(t,y) + (x^2 - z^2)c(t,y) + xzd(t,y) + zE(t,y),$$
$$\xi^4 = l(t,y) - xE(t,y) - (x^2 - z^2)\frac{1}{2}d(t,y) + xz2c(t,y) + zb(t,y).$$

The next step is to substitute this result into (4.51):

$$\begin{aligned}
&- \mathrm{e}^{\frac{2}{3}\theta t}(l(t,y)_y - xE(t,y)_y - \frac{1}{2}d(t,y)_y(x^2 - z^2) \\
&+ xz2c(t,y)_y + zb(t,y)_y) \\
&+ \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)+\frac{2}{3}\theta t}(l(t,y)_t - xE(t,y)_t - \frac{1}{2}(x^2 - z^2)d(t,y)_t \\
&+ xz2c(t,y)_t + zb(t,y)_t) \\
&- \frac{3}{2\theta}\,\mathrm{e}^{\alpha(t,x)}\,d(t,y) = 0 \\
&\Rightarrow\ l(t,y)_y = 0 \text{ with } b, c, d, E \in \mathbb{R} \\
&\Rightarrow\ \frac{1}{2}\,\mathrm{e}^{\frac{2}{3}\theta t}\,l(t)_t = \frac{3}{2\theta}d \\
&\Rightarrow\ l(t) = -\frac{9}{2\theta^2}d\,\mathrm{e}^{\frac{2}{3}\theta t} + l,\ \ l \in \mathbb{R}.
\end{aligned}$$

**Result:**
$\xi^2 = a(t,y) + bx + (x^2 - z^2)c + xzd + Ez,$
$\xi^4 = l(t) - Ex - \frac{1}{2}(x^2 - z^2)d + 2cxz + bz$ with $b, c, d, E \in \mathbb{R}$.

Similarly one obtains from (4.52):

$$\begin{aligned}
&- \frac{3}{2\theta}\,\mathrm{e}^{\alpha(t,x)}\,2c + \frac{1}{2}\,\mathrm{e}^{\alpha(t,x)+\frac{2}{3}\theta t}\,a(t,y)_t - \mathrm{e}^{\frac{2}{3}\theta t}\,a(t,y)_y = 0 \\
&\Rightarrow\ a(t,y) = a(t),\quad a(t) = -\frac{9}{\theta^2}c\,\mathrm{e}^{-\frac{2}{3}\theta t} + a,\ \ a \in \mathbb{R}.
\end{aligned}$$

From (4.36) one now obtains

$$\xi^1 = -\frac{3}{\theta}(b + 2cx + dz)$$

and from (4.35):

$$\xi^3(t,x,y,z) = \xi^3(x,y,z).$$

Now (4.39) reads as

$$\xi^3_{,2} = \mathrm{e}^{\alpha(t,x)}\,\frac{12c}{\theta}$$
$$\Rightarrow\; c = 0,\; \xi^3(x,y,z) = \xi^3(y,z).$$

**Result:**
$\xi^1 = -\frac{3}{\theta}(b + zd)$,
$\xi^2 = a + bx + xzd + Ez$,
$\xi^3 = \xi^3(y,z)$,
$\xi^4 = l(t) - Ex - \frac{1}{2}(x^2 - z^2)d + bz$.

Substituting this result into (4.41) yields

$$\xi^3_{,4} = -\,\mathrm{e}^{-\alpha(t,x)}\,-\frac{6d}{\theta}$$
$$\Rightarrow d = 0,\; \xi^3(y,z) = \xi^3(y).$$

**Result:**
$\xi^1 = -\frac{3}{\theta}b$,
$\xi^2 = a + bx + Ez$,
$\xi^3 = \xi^3(y)$,
$\xi^4 = l - Ex + bz$.

Finally from (4.37) one obtains:

$$\xi^3_{,3} = b - \sqrt{2}\omega_0(a + bx + Ez)$$
$$\Rightarrow\; E = b = 0$$
$$\Rightarrow\; \xi^3 = -\sqrt{2}\omega_0 ay + C.$$

**Result:**
$\xi^1 = 0$,
$\xi^2 = a$,
$\xi^3 = -\sqrt{2}\omega_0 ay + E$,
$\xi_4 = l$ with $a, l, C \in \mathbb{R}$.

Therefore this metric has three Killing vectors:

$$\begin{pmatrix} 0 \\ 1 \\ -\sqrt{2}\omega_0 y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using Theorem 2.2 one can now prove that there are at most three solutions. Thus at the point $p = (0,0,0,0)$ there are exactly three linearly independent solutions. The system $L_{\xi_p} R_{abcd} = 0$ contains seven linearly independent equations:

$$
\begin{aligned}
L_\xi R_{1212} &= \frac{-2}{3}\sqrt{2}\omega_0\theta\ \xi^1_{;2} + \frac{-2}{3}\sqrt{2}\omega_0\theta\ \xi^2_{;3} = 0,\\
L_\xi R_{1213} &= \frac{-1}{18}\sqrt{2}\omega_0\theta^2\ \xi^1 + \frac{1}{3}\sqrt{2}\omega_0\theta\ \xi^1_{;3} = 0,\\
L_\xi R_{1214} &= \omega_0^2\ \xi^2_{;4} + \frac{1}{3}\sqrt{2}\omega_0\theta\ \xi^3_{;4} = 0,\\
L_\xi R_{1223} &= \frac{2}{3}\sqrt{2}\omega_0\theta\ \xi^1 + \frac{1}{6}\sqrt{2}\omega_0\theta\ \xi^1_{;2} + 2\omega_0^2\ \xi^1_{;3} + \frac{1}{3}\sqrt{2}\omega_0\theta\ \xi^2_{;3} = 0,\\
L_\xi R_{1224} &= \omega_0^2\ \xi^1_{;4} + \frac{-1}{3}\sqrt{2}\omega_0\theta\ \xi^2_{;4} = 0,\\
L_\xi R_{1313} &= \frac{1}{3}\sqrt{2}\omega_0\theta\ \xi^2_{;3} = 0,\\
L_\xi R_{2334} &= \frac{-3}{2}\omega_0^2\ \xi^2_{;4} + \frac{-1}{6}\sqrt{2}\omega_0\theta\ \xi^3_{;4} = 0.
\end{aligned}
$$

The determinant of their representative matrix is

$$\frac{8}{81}\omega_0^{10}\theta^5$$

which only vanishes at $\omega_0 = 0$. Theorem 2.2 then tells us that there can be at most three solutions for $\omega_0 \neq 0$.

## 4.4 Bianchi Classification in the Gödel-type Metrics

The two metrics from section 4.2 and 4.3 have the Killing vectors

$$\begin{pmatrix} 0 \\ -\frac{1}{\omega_0} \\ y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The Gödel metric contains these vectors as a Lie subalgebra of **Bianchi type III** (see section 4.1). Thus the generalized metrics have "lost" two Killing vectors.

# Appendix A

# Sourcecode for the maple® Worksheet

This maple® worksheet calculates the representative matrix of the system of linear equations from Theorem 2.2 for a given metric. It then finds all non vanishing minors and calculates their zeros with respect to previously defined constants. In this case this is the constant $\omega_0$. The output of this worksheet is a list of minors and their zeros with respect to constants in the metric such as $\omega_0$. All that remains to do is to pick two minors from this list that do not vanish at the same points.

The command "restart" is necessary to make sure that this worksheet returns correct results.

**restart;**

The "LinearAlgebra" package is used to calculate matrices, determinants etc.

**with(LinearAlgebra):**

Now the GRTENSOR II package must be started:

**grtw():**

**GRTensorII Version 1.79 (R4)**
**6 February 2001**
**Developed by Peter Musgrave, Denis Pollney and Kayll Lake**
**Copyright 1994-2001 by the authors.**
**Latest version available from: http://grtensor.phy.queensu.ca/**

Since we want to use this worksheet for the model with vanishing acceleration and other models, the next line is optional:

```
s(t):=exp(t):
```

Now the metric needs to be included. In this case it is the metric with vanishing acceleration.

```
qload(vanacc):
```

First all objects in Remark 2.2 need to be calculated:

```
grcalc(R(dn, dn, dn, dn)), gralter(_, simplify):
grcalc(R(up, dn, dn, dn)), gralter(_, simplify):
grcalc(R(dn, up, dn, dn)), gralter(_, simplify):
grcalc(R(dn, dn, up, dn)), gralter(_, simplify):
grcalc(R(dn, dn, dn, up)), gralter(_, simplify):
grcalc(Chr(dn,dn,up)), gralter(_, simplify):
grcalc(R(dn, dn, dn, dn, cup)), gralter(_, simplify):

chris:=grarray(Chr(dn,dn,up)):

Riem :=grarray(R(dn,dn,dn,dn)):
Riem1:=grarray(R(up,dn,dn,dn)):
Riem2:=grarray(R(dn,up,dn,dn)):
Riem3:=grarray(R(dn,dn,up,dn)):
Riem4:=grarray(R(dn,dn,dn,up)):

cRiem:=grarray(R(dn,dn,dn,dn,cup)):
```

Next we define an array "koef" and fill it with zeros:
Later this array will contain the coefficients of the system of linear equations.

```
koef:=array(1..21,1..4,0..4);
for r1 from 1 to 21 do
for r2 from 1 to 4 do
for r3 from 0 to 4 do
   koef[r1,r2,r3]:=0:
end do:
end do;
end do;
```

Now we define a new procedure "proc". This procedure does the following: It assigns the term $R_{ijkl;a}$ to koef[g,a,0]. This is the term that the variable $\xi^a$ is multiplied with. Then it writes the term that $\xi_{a;i}$ is multiplied with on koef[g,a,i] (see Remark 2.4). What it also does is that it assigns every combination of integers (ijkl) to a number $g$. Since we deal with 4-dimensional manifolds this yields that $g$ runs from 1 to 21.

```
prozedur:=proc (i,j,k,l,g) local old, a: global Riem,cRiem,koef:
for a from 1 to 4 do
  koef[g,a,0]:= cRiem[i,j,k,l,a]:
  old:=koef[g,a,i]: koef[g,a,i]:=simplify(old+Riem1[a,j,k,l]):
  old:=koef[g,a,j]: koef[g,a,j]:=simplify(old+Riem2[i,a,k,l]):
  old:=koef[g,a,k]: koef[g,a,k]:=simplify(old+Riem3[i,j,a,l]):
  old:=koef[g,a,l]: koef[g,a,l]:=simplify(old+Riem4[i,j,k,a]):
end do:
end proc:

prozedur(1,2,1,2,1): prozedur(1,2,1,3,2): prozedur(1,2,1,4,3):
prozedur(1,2,2,3,4): prozedur(1,2,2,4,5): prozedur(1,2,3,4,6):
prozedur(1,3,1,3,7): prozedur(1,3,1,4,8): prozedur(1,3,2,3,9):
prozedur(1,3,2,4,10): prozedur(1,3,3,4,11): prozedur(1,4,1,4,12):
prozedur(1,4,2,3,13): prozedur(1,4,2,4,14): prozedur(1,4,3,4,15):
prozedur(2,3,2,3,16): prozedur(2,3,2,4,17): prozedur(2,3,3,4,18):
prozedur(2,4,2,4,19): prozedur(2,4,3,4,20): prozedur(3,4,3,4,21):
```

The next step subtracts the factor of $\xi_{i;a}$ from the factor of $\xi_{a;i}$ for $a < i$ and assigns the result to koef[g,a,i].

```
for g from 1 to 21 do
  old:=koef[g,1,2]: koef[g,1,2]:=old-koef[g,2,1]:
  old:=koef[g,1,3]: koef[g,1,3]:=old-koef[g,3,1]:
  old:=koef[g,1,4]: koef[g,1,4]:=old-koef[g,4,1]:
  old:=koef[g,2,3]: koef[g,2,3]:=old-koef[g,3,2]:
  old:=koef[g,2,4]: koef[g,2,4]:=old-koef[g,4,2]:
  old:=koef[g,3,4]: koef[g,3,4]:=old-koef[g,4,3]:
end do:
```

Since now only the coefficients of $\xi_{a;i}$ for $a < i$ are of interest we assign them to a matrix "m".

```
m:=Matrix(21,10):

for g from 1 to 21 do
  m[g,1]:=koef[g,1,0]:
  m[g,2]:=koef[g,2,0]:
```

```
  m[g,3]:=koef[g,3,0]:
  m[g,4]:=koef[g,4,0]:
  m[g,5]:=koef[g,1,2]:
  m[g,6]:=koef[g,1,3]:
  m[g,7]:=koef[g,1,4]:
  m[g,8]:=koef[g,2,3]:
  m[g,9]:=koef[g,2,4]:
  m[g,10]:=koef[g,3,4]:
end do:
```

Next we determine the point $p = (t, x, y, z)$ where the equations $R_{ijkl;a} = 0$ should be evaluated.

```
t:=0; x:=0: y:=0: z:=0:
```

The last step is that we calculate all minors of the matrix "m". Since we want to avoid redundant calculations we only calculate determinants of matrices where columns and lines are in ascending order.

```
for r1 from 1 to 15 do
for r2 from r1+1 to 16 do
for r3 from r2+1 to 17 do
for r4 from r3+1 to 18 do
for r5 from r4+1 to 19 do
for r6 from r5+1 to 20 do
for r7 from r6+1 to 21 do

for s1 from 1 to 4 do
for s2 from s1+1 to 5 do
for s3 from s2+1 to 6 do
for s4 from s3+1 to 7 do
for s5 from s4+1 to 8 do
for s6 from s5+1 to 9 do
for s7 from s6+1 to 10 do

   minor:=simplify(Determinant(SubMatrix(m,[r1,r2,r3,
   r4,r5,r6,r7], [s1,s2,s3,s4,s5,s6,s7])));
   if minor<>0 then
         print(r1,r2,r3,r4,r5,r6,r7,s1,s2,s3,s4,s5,s6,s7);
         print(minor): zeroes:=solve(minor=0): print(zeroes);
   end if;
end do; end do; end do; end do; end do; end do; end do;
end do; end do; end do; end do; end do; end do; end do;
```

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